Mathematics

Fairfield Public Schools

Multivariable Calculus

AP CALCULUS BC

Critical Areas of Focus

Multivariable Calculus is a rigorous second year course in college level calculus. This course provides an in-depth study of vectors and the calculus of several variables for the student who has successfully completed AP Calculus. The successful student will bring to the course a solid understanding of the concepts of first-year calculus as well as the ability to approach complex problems and applications with insight, imagination, and persistence. The critical areas of focus for this course are:

1) The dot product and cross product of vectors are given geometric definitions, motivated by work and torque, before the algebraic expressions are deduced. To facilitate the discussion of surfaces, functions of two variables and their graphs are introduced.

2) The calculus of vector functions is used to provide Kepler's First Law of planetary motion. In keeping with the introduction of parametric curves as introduced in prior courses, parametric surfaces are introduced, thus allowing for a discussion of tangent planes and areas of parametric surfaces.

3) Functions of two or more variables are studied from verbal, numerical, visual, and algebraic points of view. Directional derivatives are estimated from contour maps of temperature, pressure, and snowfall.

4) Contour maps and the Midpoint Rule are used to estimate the average snowfall and average temperature in given regions. Double and triple integrals are used to compute probabilities, areas of parametric surfaces, volumes of hyper-spheres, and the volume of intersection of three cylinders.

Pacing Guide									
1st Marking Period		2nd Marking Period			3rd Marking Period			4th Marking Period	
September	October	November	December	January	February	March	April	May	June
Unit 1 **Vectors and the Geometry of Space** 9 weeks		Unit 2 **Vector Functions** 9 weeks			Unit 3 **Partial Derivatives** 9 weeks			Unit 4 **Multiple Integrals** 9 weeks	

Course Overview		
Central Understandings Insights learned from exploring generalizations through the essential questions. (Students will understand) • We live in a three dimensional world and in order to understand that world, physicist and mathematicians need models that involve multi-variables.	**Essential Questions** • How can we use the concepts of multivariable calculus to understand the physical world around us?	**Assessments** • Formative Assessments • Summative Assessments

Content Outline	Standards
I. Unit 1 – Vectors and the Geometry of Space II. Unit 2 – Vector Functions III. Unit 3 – Partial Derivatives IV. Unit 4 – Multiple Integrals	The standards referenced in this curriculum are from the Georgia Multivariable Calculus standards.

Calculus Standards for Mathematical Practice

The K-12 Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. This page gives examples of what the practice standards look like at the specified grade level. Students are expected to:

Standards	***Explanations and Examples***
1. Make sense of problems and persevere in solving them.	Students solve problems involving equations and discuss how they solved them. Students solve real world problems through the application of algebraic and geometric concepts. Students seek the meaning of a problem and look for efficient ways to represent and solve it. They may check their thinking by asking themselves, "What is the most efficient way to solve the problem?", "Does this make sense?", and "Can I solve the problem in a different way?"
2. Reason abstractly and quantitatively.	This practice standard refers to one of the hallmarks of algebraic reasoning, the process of de-contextualization and contextualization. Much of elementary algebra involves creating abstract algebraic models of problems and then transforming the models via algebraic calculations to reveal properties of the problems.
3. Construct viable arguments and critique the reasoning of others.	In Calculus, students construct arguments using verbal or written explanations accompanied by expressions, equations, inequalities, models, graphs and tables. They further refine their mathematical communication skills through mathematical discussions in which they critically evaluate their own thinking and the thinking of other students. They pose questions like "How did you get that?", "Why is that true?" "Does that always work?" They explain their thinking to others and respond to others' thinking.
4. Model with mathematics.	Indeed, other mathematical practices in Calculus might be seen as contributing specific elements of these two. The intent of the following set is not to decompose the above mathematical practices into component parts but rather to show how the mathematical practices work together.
5. Use appropriate tools strategically.	Students consider available tools such as spreadsheets, a function modeling language, graphing tools and many other technologies so they can strategically gain understanding of the ideas expressed by individual content standards and to model with mathematics.
6. Attend to precision.	In Calculus, the habit of using precise language is not only a mechanism for effective communication but also a tool for understanding and solving problems. Describing an idea precisely helps students understand the idea in new ways.
7. Look for and make use of structure.	In Calculus, students should look for various structural patterns that can help them understand a problem. For example, seeing the pattern to the power rule after using the derivative with limits (e.g., $f(x) = 3x^2$ results in $f'(x) = 6x$ after $\lim_{h\to o} \frac{3(x+h)^2-3x^2}{h}$).
8. Look for and express regularity in repeated reasoning.	Creating equations or functions to model situations is harder for many students than working with the resulting expressions. An effective way to help students develop the skill of describing general relationships is to work through several specific examples and then express what they are doing with algebraic symbolism.

Unit 1 – Vectors and the Geometry of Space, 9 weeks top

In this unit, the concepts of vectors and coordinate systems for three-dimensional space are developed. This is the setting for the study of functions of two variables because the graph of such a function is a surface in space. Vectors provide particularly simple descriptions of lines and planes in space as well as velocities and accelerations of objects that move in space. Students will investigate the dot product and cross product of two vectors; define relationship among points, lines, and planes in three dimensions; understand and apply properties of matrices and determinants including Cramer's Rule and Gaussian Elimination.

Big Ideas The central organizing ideas and underlying structures of mathematics	**Essential Questions**
• Vectors and coordinate systems for three-dimensional space. • Real-Value functions of two variables as a representation of a surface in space • Vectors as a way of describing lines and planes in space as well as velocities and accelerations of objects that move through space.	• What is a vector, what are its properties and how does one measure its magnitude and direction? • What operations can be performed using vectors? • What is a dot product and how can it be used to measure the work done by a force? • What is a cross product and what are its applications in physics and engineering? • What is a determinant? • How can determinants be used to solve systems of equations involving two or more variables (Cramer's Rule)? • What is Gaussian Elimination and how can it be used to solve systems of equations involving multiple variables? • How are lines and planes defined in three space? • How can you use vectors to determine equations of lines and planes in space? • How are functions in two variables defined and how do they relate to surfaces in three space? • How do the rectangular, cylindrical and spherical coordinate systems relate to each other in three-space.

Calculus Standards

Investigate the relationship between points, lines, and planes in three-dimensions.

MVC.3-D.1
Represent equations of lines in space using vectors.

MVC.3-D.2
Express analytic geometry of three dimensions (equations of planes, parallelism, perpendicularity, angles) in terms of the dot product and cross product of vectors.

Recognize and apply properties of matrices.

MVC.M.1

Find the determinant of 2-by-2 and 3-by-3 matrices.

MVC.M.2

Represent a 3-by-3 system of linear equations as a matrix and solve the system in multiple ways: the inverse matrix, row operations, and Cramer's Rule.

MVC.M.3

Apply properties of similar and orthogonal matrices to prove statements about matrices.

MVC.M.4

Find and apply the eigenvectors and eigenvalues of a 3-by-3 matrix.

Unit 2 – Vector Functions, 9 weeks [top]

The functions the students studied so far have been around real-valued functions. The focus of this unit shifts to functions whose values are vectors because such functions are needed to describe curves and surfaces in space. This unit will also include vector-valued functions to describe the motion of objects through space. In particular, the use of vector functions will allow for the development of Kepler's laws of planetary motion.

Big Ideas The central organizing ideas and underlying structures of mathematics	**Essential Questions**
• Vector-valued functions as a description of curves and surfaces in space. • Vector-valued functions as a description of the motion of objects through space.	• What is a vector function? • What is a derivative/integral of a vector function? • What is a space curve and how do we measure its length and curvature? • What are tangent, bi-normal, and normal vectors as well as the normal and osculating planes and how do they relate to a space curve? • How can the ideas of a tangent and normal vectors and curvature be used in physics to study the motion of an object, including its velocity and acceleration, along a space curve? • What is a parametric surface and how are they defined by parametric vector functions?

Calculus Standards

Explore functions of two independent variables of the form *z = f(x, y)* and implicit functions of the form *f(x, y, z) = 0*.

MVC.E-F.1
Evaluate such functions at a point in the plane.

MVC.E-F.2
Graph the level curves of such functions.

MVC.E-F.3
Determine points or regions of discontinuity of such functions.

Unit 3 – Partial Derivatives, 9 weeks top	
Physical quantities often depend on two or more variables. In this unit, students will extend the basic ideas of differential calculus to such functions. Students will investigate limits, continuity, and differentiation of functions of two independent variables; define and apply the gradient, the divergence, and the curl.	
Big Ideas The central organizing ideas and underlying structures of mathematics	**Essential Questions**
• The concepts of differential calculus as they relate to various real and vector valued functions. • Various functions will be explored and understood verbally, numerically, algebraically and visually	• What is a contour map and what are level curves and level surfaces? • What does it mean for a function in three-space to be continuous? • How can you determine if a limit exists in three-space? • What is a partial derivative and how is it interpreted? • What is Laplace's equation and the wave equation and what are there significance? • What is a linear or tangent plane approximation of a function at a point and what is it used for? • How is the chain rule applied when taking derivatives of functions of two variables? • How does one implicitly differentiate a function of three variables? • What is the Implicit Function Theorem? • What is a directional derivative? • What is a gradient vector and what meaning does it have? • How does one calculate the minima and maxima values of a function of two variables? • What applications are there for maximizing or minimizing the value of a function?
Calculus Standards	
Explore the continuity of functions of two independent variables in terms of the limits of such functions as (*x*, *y*) approaches a given point in the plane. **MVC.FC.1** Determine the continuity of a function given a point on the plane. **Explore, find, use, and apply partial differentiation of functions of two independent variables of the form $z = f(x, y)$ and implicit functions of the form $f(x, y, z) = 0$.** **MVC.PD.1** Approximate the partial derivatives at a point of a function defined by a table of data. **MVC.PD.2** Find expressions for the first and second partial derivatives of a function.	

MVC.PD.3
Define and apply the total differential to approximate real-world phenomena.
MVC.PD.4
Represent the partial derivatives of a system of two functions in two variables using the Jacobian.
MVC.PD.5
Find the partial derivatives of the composition of functions using the general chain rule.
MVC.PD.6
Apply partial differentiation to problems of optimization, including problems requiring the use of the Lagrange multiplier.

Unit 4 – Integration, 9 weeks top

In this unit, the idea of a definite integral is extended to double and triple integrals of function of two or three variables. These concepts of double and triple integrals are then used to compute volumes, surface areas, masses, and centroids of more general regions than we were able to consider in prior units. Students will explore double and triple integrals and integrals of vectors; use various methods of integration; understand and apply the theorems of Green, Stokes, and Gauss.

Big Ideas The central organizing ideas and underlying structures of mathematics	**Essential Questions**
• The idea of the definite integral is extended to double and triple integrals of functions of two and three variables • Double and Triple integrals are used to calculate volumes, surface areas, masses, electrical charges and other variable characteristics of a surface.	• What is a double integral and how can it be used to find the volume of a solid? • How do you express a double integral as an iterated integral so that we can use standard integration methods to evaluate the expression? • How do we integrate a function of a general, non-rectangular, region? • How do you use polar coordinates to simplify the integration of solids over circular regions? • How can double integrals be used to calculate mass, electrical charge, center of mass, moment of inertia and other physical attributes of a solid? • How can double integrals be used to calculate the surface area of a solid? • What is a triple integral and how can it be used to evaluate functions of three variables? • How can triple integrals be used to calculate various physical attributes of a function of three variables, such as density? • How do we evaluate the triple integral of certain solids using cylindrical or spherical coordinates? • How can the method of substitution, change of variables, be used to simplify an integral? • What is a Jacobian?

Calculus Standards

Integrate functions of the form $z = f(x, y)$ or $w = f(x, y, z)$.

MVC.IF.1
Define, use, and interpret double and triple integrals in terms of volume and mass.

MVC.IF.2
Represent integrals of vectors as double and triple integrals.

MVC.IF.3
Integrate functions through various techniques such as changing the order of integration, substituting variables, or changing to polar coordinates.

Apply and interpret the theorems of Green, Stokes, and Gauss.

MVC.T.1
Apply line and surface integrals to functions representing real-world phenomena.

MVC.T.2
Recognize, understand, and use line integrals that are independence of path.

MVC.T.3
Define and apply the gradient, the divergence, and the curl in terms of integrals of vectors.