# AP Calculus Summer Prep

## Topics from Algebra and Pre-Calculus

(Solutions are the last pages of this handout)

The purpose of this packet is to give you a review of basic skills. In calculus you will need to recall every math concept you have ever learned😉, but this packet will recall the most recent you have learned and those topics that will show up in the beginning of the year.

You are asked to have this packet completed for the first day of school and **all problems are expected to be completed without the use of a graphing calculator**. Questions will be answered on the first day of class.

## ALGEBRA TOPICS

FACTORING POLYNOMIALS AND RATIONALS: basic factoring and factoring of binomials, fractional exponents and binomials within rational expressions to simplify.

Factor completely:

1) $25x^2 - 16y^2$

2) $3e^{2x} - 5e^x - 2$

3) $\frac{1}{2}x^3(x+4)^5 - 2x^2(x+4)^6$

4) $2x^{\left(\frac{1}{2}\right)}(x+2) + 2x^{\left(\frac{3}{2}\right)}(x+2)^3$

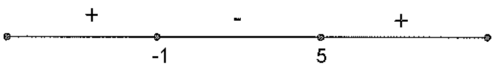
5) $\frac{8x(x+5)^3 - 4x^2(x+5)^2}{16x^2+80x}$

6) $\frac{3(2x^2-8)+12(x+2)^2}{12x^2+6x-36}$

## SOLVING POLYNOMIAL INEQUALITIES

Example 1: $x^2 - 4x - 5 < 0$

Solution: $(x+1)(x-5) < 0$ *factor and determine critical values: x = -1, 5*



Answer: (-1, 5) *mark the zeros; pick a test point to determine the sign of the polynomial in each interval- this is called a SIGN CHART!*

PRACTICE PROBLEMS – Include a Sign Chart

7) $x^3 + 7x^2 + 10x > 0$

8) $3x(x+2)^3 + 9x^2(x+2)^2 \le 0$

# PRE-CALCULUS TOPICS

**Analyzing Graphs:**

Domain and Range, x-intercepts (zeros) and y-intercepts, extrema (local and absolute)

End Behavior (limits at infinity): $\lim_{x \to \pm\infty} f(x)$

Continuity and Discontinuity: All Polynomials are continuous for all x. Types of discontinuities (removable, removable with point, infinite and jump).

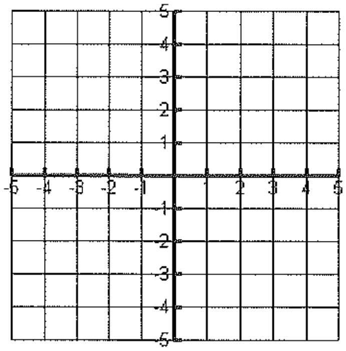
Even/Odd Functions:

Even: Symmetric to the y-axis. Algebraically: $f(-x) = f(x)$

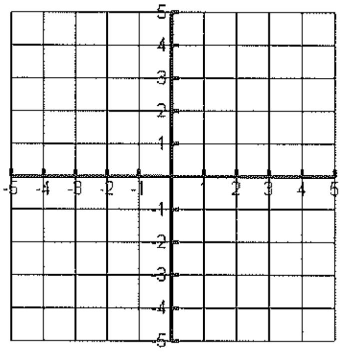
Odd: Symmetric to the origin. Algebraically: $f(-x) = -f(x)$

Parent Functions that all graduating Pre-Calculus students should be able to sketch and identify: Sketch the following parent functions. Determine domain, interval of continuity, symmetry, end behavior and type and location of any discontinuities.

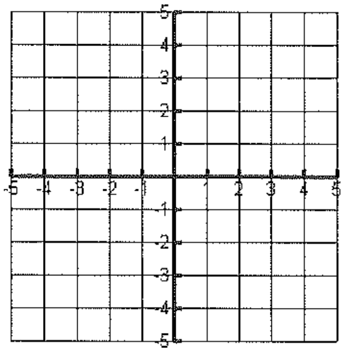
Linear: $y = x$



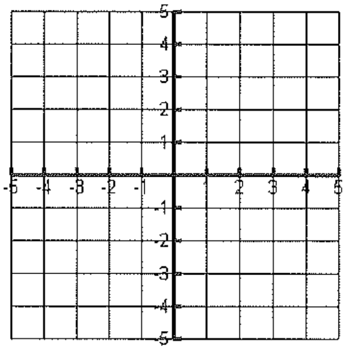
Quadratic: $y = x^2$



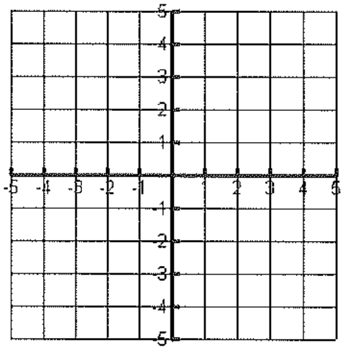
Cubic: $y = x^3$



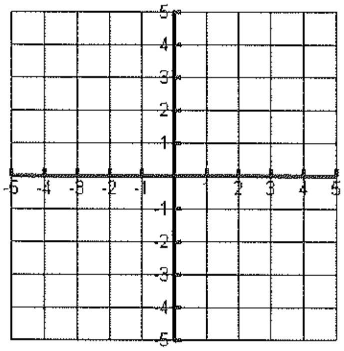
Absolute Value: $y = |x| = \begin{cases} x, if\ x \geq 0 \\ -x, if\ x < 0 \end{cases}$



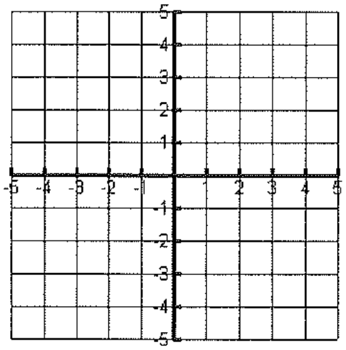
Rational: $y = \frac{1}{x}$



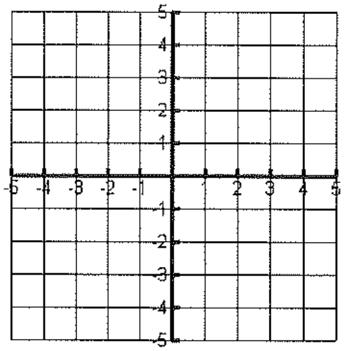
Root: $y = \sqrt{x}$



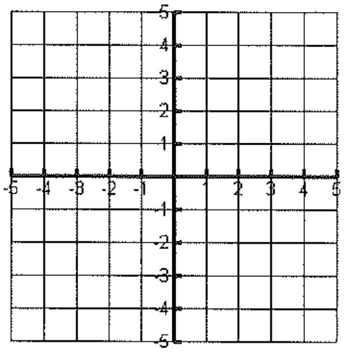
Exponential Growth: $y = e^x$ where $x > 0$



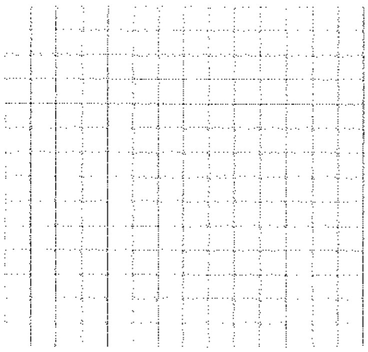
Exponential Decay: $y = e^x$ where $x < 0$



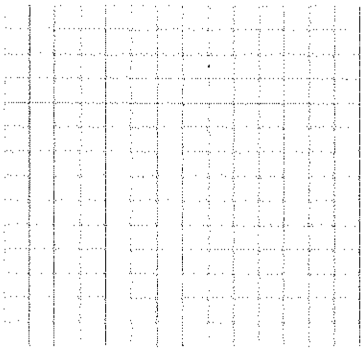
Logarithmic Growth: $y = \ln x$



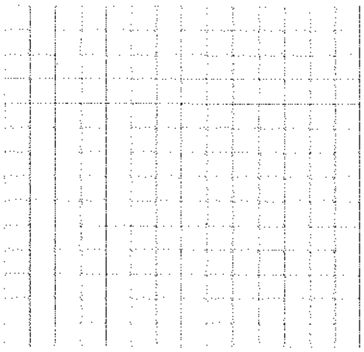
Sine: $y = \sin(x)$



Cosine: $y = \cos(x)$



Tangent: $y = \tan(x)$



Analyze the following functions without graphing them on a graphing calculator (except where noted).

9) $f(x) = x^2 - 2x - 3$

Domain:

Range:

x-intercept(s):

y-intercept(s):

Extrema (by hand):

End Behavior:

$\lim_{x \to -\infty} f(x) =$

$\lim_{x \to \infty} f(x) =$

Interval of Continuity:

Tests for Symmetry:

10) $g(x) = 6(x + 2)^2 - 4(x + 2)^3$

Domain:

Range:

x-intercept(s):

y-intercept(s):

End Behavior:

$\lim_{x \to -\infty} g(x) =$

$\lim_{x \to \infty} g(x) =$

Interval of Continuity:

**Rational Expressions and Functions:**
Domain and Range
End Behavior (limits at infinity): $\lim_{x\to\pm\infty} f(x) = horizontal\ and\ Oblique\ asymptote$
Discontinuity: infinite discontinuities occur at vertical Asymptotes and removable discontinuities occur at holes.

11) Determine the following for the function $f(x) = \frac{3x^2+5x+2}{2x^2+x-1}$

Domain:

Range:

x-intercept(s):

y-intercept(s):

End Behavior:

$\lim_{x\to-\infty} f(x) =$

$\lim_{x\to\infty} f(x) =$

Interval of Continuity:

Name and Location of Discontinuities:

12) Simplify

a. $\frac{3e^x x^2-3e^x}{e^{4x}(x+1)^3}$

b. $\frac{3x^{-2}}{1x^{-1}-\frac{1}{3}x^{-4}}$

c. $\frac{\frac{1}{x+1}}{\frac{2}{x-3}-\frac{1}{x^2-2x-3}}$

13) Solve

a. $\frac{x+3}{x^2-2x-8} - \frac{x-5}{x^2-12x+32} \leq 0$

b. $\frac{5x(x^2+1)(x+2)^{-1}+10(x^2+1)}{20(x+2)(x^2+1)^3} \geq 0$

## FUNCTION OPERATIONS AND COMPOSITION; INVERSE FUNCTIONS

**Composition** of a function $g$ with a function $f$ is defined as:

$$h(x) = g(f(x))$$

The domain of h is the set of all x-values such that x is in the domain of f and f(x) is in the domain of g.

**Inverses:** Functions f and g are inverses of each other provided:

$$f(g(x)) = x \quad and \quad g(f(x)) = x$$

The function g is denoted as $f^{-1}$, read as "f inverse".

**Horizontal Line Test:** If any horizontal line which is drawn through the graph of a function f intersects the graph no more than once, then f is said to be a **one-to-one** function and has an inverse.

14) Let f and g be functions whose values are given by the table below. Assume g is one-to-one with inverse $g^{-1}$.

a. $f(g(3))$

b. $g^{-1}(4)$

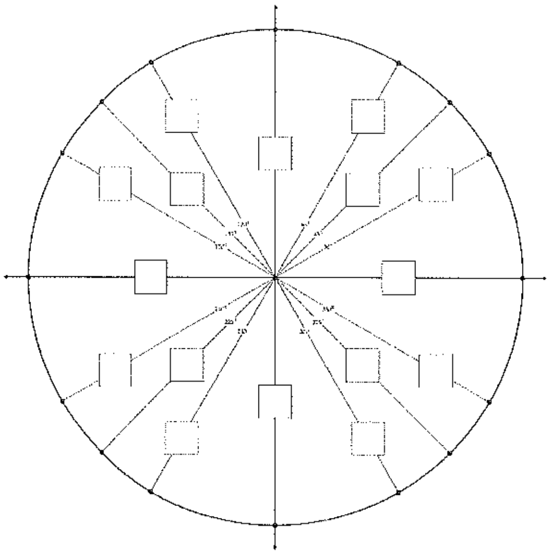
c. $f(g^{-1}(6))$

d. $f^{-1}\left(f(g(2))\right)$

| X | f(x) | g(x) |
|---|---|---|
| 1 | 6 | 2 |
| 2 | 9 | 3 |
| 3 | 10 | 4 |
| 4 | -1 | 6 |

## TRIGONOMETRY

UNIT CIRCLE: Complete angles and Sine/Cosine



Key Trig Identities - formulas you should know:

$$\sin^2 x + \cos^2 x = 1$$
$$\sin(2x) = 2\sin x \cos x$$
$$\cos(2x) = 1 - 2\sin^2 x$$
$$1 + \tan^2 x = \sec^2 x$$
$$\sin(-x) = -\sin x \quad ODD$$
$$\cos(-x) = \cos x \quad EVEN$$

15) Without using a calculator or table, find each value:

a. $\cos\left(-\frac{\pi}{3}\right)$

b. $\sec\left(\frac{11\pi}{6}\right)$

c. $\tan^{-1}(-1)$

d. $\sec^{-1}(-2)$

16) Solve the trigonometric equations algebraically by using identities and **without** the use of a calculator. Find all solutions in the interval $0 \le \theta \le 2\pi$.

a. $2\sin^2\theta - 1 = 0$

b. $\sin(2x) = \cos x$

c. $2\cos^2\theta + \cos\theta - 1 = 0$

## EXPONENTIAL AND LOGARITHMIC FUNCTIONS:

**Logarithm:** For any positive numbers b and y with $b \neq 1$, we define the **logarithm of y with base b** as follows: $\log_b y = x$ if and only if $b^x = y$

**LAWS OF LOGARITHMS** (for M, N, b $> 0$, $b \neq 1$)

$(i)\ \log_b MN = \log_b M + \log_b N$

$(ii)\ \log_b \frac{M}{N} = \log_b M - \log_b N$

$(iii)\ \log_b M = \log_b N \text{ if and only if } M = N$

$(iv)\ \log_b M^k = k \cdot \log_b M$

$(v)\ b^{\log_b M} = M$

The logarithmic and exponential functions are **inverse** functions.

Example: Consider $f(x) = 2^x$ and $g(x) = \log_2 x$. Verify that (3,8) is on the graph of f and (8,3) on the graph of g. In addition, $f(g(x)) = 2^{\log_2 x} = x$ *and* $g(f(x)) = \log_2(2^x) = x$ which verifies that f and g are indeed inverse functions.

Simplify.

17) $\log_5 \frac{1}{125}$

18) $\log_4 \sqrt[8]{16}$

19) $2\ln\frac{3}{e^3}$

20) $\frac{1}{t}\ln e^t$

21) $e^{3\ln(x+5)}$

22) $e^{-lnx}$

23) $4\ln e^{x^2}$

24) $e^{3x+ln5}$

25) $e^{3(\ln(x)+\ln 5)}$

26) Write $5\,ln\,(x) - 3\,ln\,(y)\ +\ 2\,ln\,(4x) - 6\,ln\,(y^2)$ as a a single logarithm.

27) Expand $\ln\left(\frac{3x^2}{2\sqrt{y^2-4}}\right)$.

28) Evaluate:

a. $\ln 1$ b. $\ln 3e$ c. $\ln e$ d. $\ln 0$ e. $\ln(\ln e)$

Limits: $\lim_{x\to c} f(x) = N$ "Limit as x approaches c of f(x) equals N"

## LIMIT ≠ CONTINUITY

Understanding the difference between limits and continuity: Limit is the value (y) that the function APPROACHES as you get close to c (either from one side or from both sides). Continuity implies that the limit exists (from both sides) and that the limit = the value at c!

**Limit Exists:** $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) \therefore \lim_{x\to c} f(x)\ exists$ and will equal all the same value or $\pm\infty$.

**Continuous:** $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)\ AND = f(c) = A\ (where\ A\ is\ a\ constant\ not\ \pm\infty)$ **MUST SHOW ALL 3!!**

**One-Sided limit**: exists if graph approaches a value from one side: + (right) or – (left)

**Evaluating Limits**:

1. **Tables:** either from table or by graphing and viewing table looking for values of y approaching the same number. Check one sided limits first and if they are equal the overall limit exists.
2. **Graphically**: Check one sided limits (again – value of limit is the y-value or the height of the graph) and if they are equal overall limit exists.
3. **Algebraically – Direct substitution:** plug in c. If you get a constant then limit exists. Rational functions may need to be simplified first!

   When you use Direct substitution and you get
   - $\frac{0}{number}$ then the limit is equal to 0
   - $\frac{0}{0}$ most likely a removable discontinuity, try simplifying or rationalizing.
   - $\frac{number}{0}$ most likely an infinite discontinuity, look at each side and evaluate one sided limit or if asking for two sided limit make sure the function is approaching the same from both sides.

$\lim_{x\to\pm\infty} f(x)$: **End Behavior:**

- Polynomial: use degree (even or odd) and sign of leading coefficient to decide parabolic or cubic.
- Rational: end behavior is determined by the Horizontal or Oblique Asymptotes.
- Other "Known" Graphs: know

Examples: Evaluate the limits:

29) $\lim_{x\to 3}(2x^2)$

30) $\lim_{x\to -2^+}\frac{1}{x^2-4}$

31) $\lim_{x\to\frac{\pi}{2}}(x\sin x)$

32) $\lim_{x\to 5}\frac{3}{x-5}$

33) $\lim_{x\to 0^+}\ln(x)$

34) $\lim_{x\to -\frac{\pi}{2}}\tan(x)$

35) $\lim_{x\to -\infty} e^{-x}$

36) $\lim_{x\to 1}\frac{3x^2-2x-1}{x^2-1}$

37) If $f(x) = \begin{cases} 2x-3, & if\ x < -1 \\ 2, & if\ x = -1 \\ 5x, & if\ x > -1 \end{cases}$, determine the following:

a. $\lim_{x\to -1^-} f(x)$

b. $\lim_{x\to -1^+} f(x)$

c. $\lim_{x\to -1} f(x)$

d. Is f continuous?

38) If $f(x) = |x+2| = \begin{cases} x+2, & x \geq -2 \\ -(x+2), & x < -2 \end{cases}$, determine the following:

a. $\lim_{x\to -2^-} f(x)$

b. $\lim_{x\to -2^+} f(x)$

c. $\lim_{x\to -2} f(x)$

d. Is f continuous?

39) Determine whether $f(x)$ is continuous. Justify. Then, determine domain, interval of continuity and location and type of any discontinuities.

a. $f(x) = \begin{cases} 3x^2, & x < 0 \\ 4, & x = 0 \\ 3\sin^2 x, & x > 0 \end{cases}$

b. $f(x) = \begin{cases} \ln x, & x < e^2 \\ \sqrt{x}, & x \geq e^2 \end{cases}$

c. $f(x) = \begin{cases} 3x^2-1, & x \leq 2 \\ 5x+1, & x > 2 \end{cases}$

40) If $\lim_{x\to c} f(x) = -2$ and $\lim_{x\to c} g(x) = 5$, determine the following:

a. $\lim_{x\to c} 5f(x)$

b. $\lim_{x\to c}[f(x) - 3g(x)]$

c. $\lim_{x\to c}\frac{(f(x))^3}{\sqrt[3]{g(x)}}$

41) Let f(x) be the graph below. Determine the following:

a. $\lim_{x \to -3^-} f(x)$

b. $\lim_{x \to -3^+} f(x)$

c. $\lim_{x \to -3} f(x)$

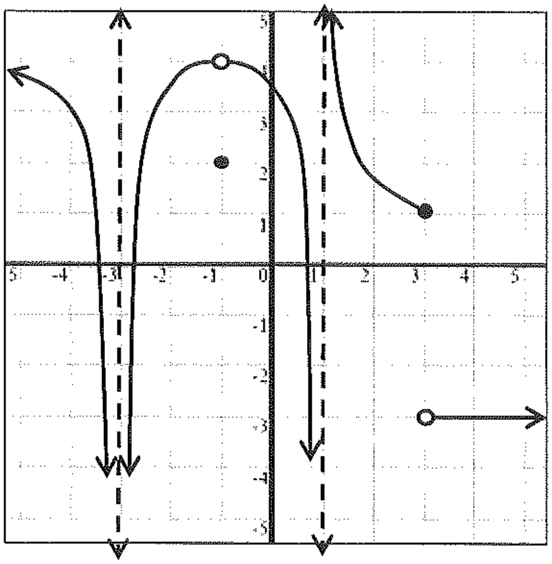
d. f(-3)

e. Is f continuous at -3?

f. $\lim_{x \to -1^-} f(x)$

g. $\lim_{x \to -1^+} f(x)$

h. $\lim_{x \to -1} f(x)$



i. f(-1)

j. $\lim_{x \to 1^-} f(x)$

k. $\lim_{x \to 1^+} f(x)$

l. $\lim_{x \to 1} f(x)$

m. $f(1)$

n. $\lim_{x \to 3^-} f(x)$

o. $\lim_{x \to 3^+} f(x)$

p. $\lim_{x \to 3} f(x)$

q. $f(3)$

r. $\lim_{x \to -\infty} f(x)$

s. $\lim_{x \to \infty} f(x)$

t. Continuity Interval:

u. Location and type of each discontinuity:

v. Domain:

42) Draw a graph given the following conditions:

- ◆ $f(0) = 0$
- ◆ $f(1) = 2$
- ◆ $f(4) = -3$
- ◆ $\lim_{x \to 3^-} f(x) = \infty$
- ◆ $\lim_{x \to -\infty} f(x) = -1$
- ◆ $\lim_{x \to \infty} f(x) = 1$
- ◆ $\lim_{x \to 3^+} f(x) = -\infty$

43) Draw a graph given the following conditions:

- ◆ $f(-2) = 0$
- ◆ $f(2) = 2$
- ◆ $\lim_{x \to -2^-} f(x) = 4$
- ◆ $\lim_{x \to -\infty} f(x) = -\infty$
- ◆ $\lim_{x \to \infty} f(x) = 5$
- ◆ $\lim_{x \to -2^+} f(x) = 0$

## ALGEBRA TOPICS

FACTORING POLYNOMIALS AND RATIONALS: basic factoring and factoring of binomials, fractional exponents and binomials within rational expressions to simplify.

Factor completely:

1) $25x^2 - 16y^2$

$(25x+4y)(25x-4y)$

2) $3e^{2x} - 5e^x - 2$

$(3e^x+1)(e^x-2)$

3) $\frac{1}{2}x^3(x+4)^5 - 2x^2(x+4)^6$

$x^2(x+4)^5\left[\frac{1}{2}x - 2(x+4)\right]$

$x^2(x+4)^5\left(-\frac{3}{2}x - 8\right)$

4) $2x^{\left(\frac{1}{2}\right)}(x+2) + 2x^{\left(\frac{3}{2}\right)}(x+2)^3$

$2x^{1/2}(x+2)\left[1 + x(x+2)^2\right]$

$2x^{1/2}(x+2)\left(1 + x(x^2+4x+4)\right)$

$2x^{1/2}(x+2)(1 + x^3+4x^2+4x) \rightarrow$

$2x^{1/2}(x+2)(x^3+4x^2+4x+1)$

$2x^{1/2}(x+2)(x+1)(x^2+3x+1)$

$-1 | \; 1 \quad 4 \quad 4 \quad 1$

$\downarrow \; -1 \quad -3 \quad -1$

$1 \quad 3 \quad 1 \;|\; 0$

5) $\frac{8x(x+5)^3-4x^2(x+5)^2}{16x^2+80x}$

$\frac{\cancel{4}\cancel{x}(x+5)^{\cancel{2}}[2(x+3)-x]}{4\,\cancel{16}\cancel{x}\cancel{(x+5)}}$

$\frac{(x+5)(x+6)}{4} = \frac{1}{4}(x+5)(x+6)$

6) $\frac{3(2x^2-8)+12(x+2)^2}{12x^2+6x-36}$

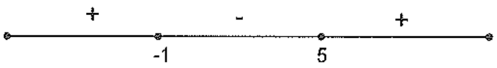
$\frac{3\cdot2(x^2-4)+12(x+2)^2}{6(2x^2+x-6)} = \frac{\cancel{6}\cancel{(x+2)}\left[(x-2)+2(x+2)\right]}{\cancel{6}(2x-3)\cancel{(x+2)}}$

$= \frac{3x+2}{2x-3}$

## SOLVING POLYNOMIAL INEQUALITIES

Example 1: $x^2 - 4x - 5 < 0$

Solution: $(x+1)(x-5) < 0$ *factor and determine critical values: x = -1, 5*



Answer: (-1, 5) *mark the zeros; pick a test point to determine the sign of the polynomial in each interval- this is called a* ***SIGN CHART!***

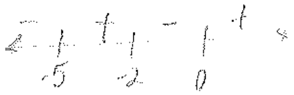
PRACTICE PROBLEMS - Include a Sign Chart

7) $x^3 + 7x^2 + 10x > 0$

$x(x^2+7x+10) > 0$

$x(x+5)(x+2) > 0$

C.V = 0, -5, -2



$(-5,-2) \cup (0,\infty)$

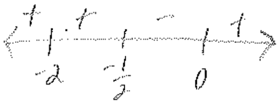
8) $3x(x+2)^3 + 9x^2(x+2)^2 \le 0$

$3x(x+2)^2[(x+2)+3x] \le 0$

$3x(x+2)^2[4x+2] \le 0$

$6x(x+2)^2(2x+1) \le 0$

C.V, x = 0, -2, -1/2



$\{-2\} \cup [-1/2, 0]$

## PRE-CALCULUS TOPICS

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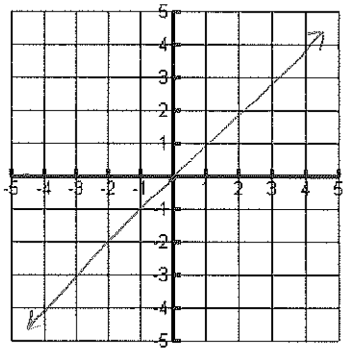
Domain and Range, x-intercepts (zeros) and y-intercepts, extrema (local and absolute)
End Behavior (limits at infinity): $\lim_{x\to\pm\infty} f(x)$

Continuity and Discontinuity: All Polynomials are continuous for all x. Types of discontinuities (removable, removable with point, infinite and jump).
Even/Odd Functions:
Even: Symmetric to the y-axis. Algebraically: $f(-x) = f(x)$
Odd: Symmetric to the origin. Algebraically: $f(-x) = -f(x)$

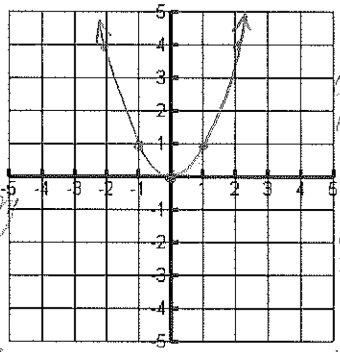
Parent Functions that all graduating Pre-Calculus students should be able to sketch and identify: Sketch the following parent functions. Determine domain, interval of continuity, symmetry, end behavior and type and location of any discontinuities.

Linear: $y = x$



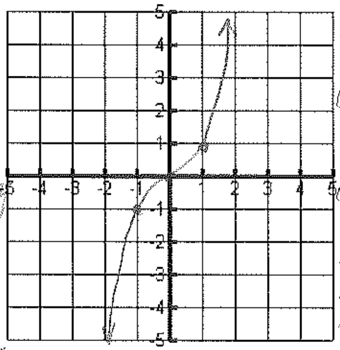
Domain: $(-\infty,\infty)$
Continuity interval $(-\infty,\infty)$
Odd Symmetry
$\lim_{x\to-\infty} y = -\infty$
$\lim_{x\to\infty} y = \infty$
no discontinuities

Quadratic: $y = x^2$



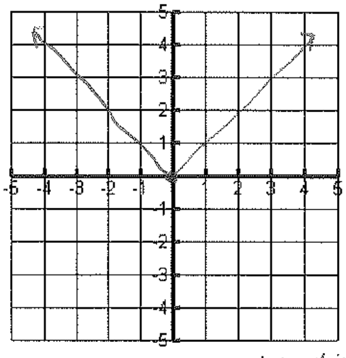
Domain $(-\infty,\infty)$
Continuity Interval: $(-\infty,\infty)$
Odd Symmetry
$\lim_{x\to-\infty} y = \infty$
$\lim_{x\to\infty} y = \infty$
no discontinuities

Cubic: $y = x^3$



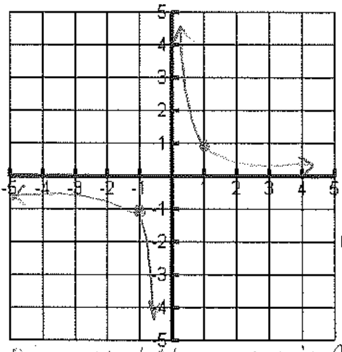
Domain: $(-\infty,\infty)$
Continuity Interval $(-\infty,\infty)$
Odd Symmetry
$\lim_{x\to-\infty} y = -\infty$
$\lim_{x\to\infty} y = \infty$
No discontinuities

Absolute Value: $y = |x| = \begin{cases} x, if\ x \geq 0 \\ -x, if\ x < 0 \end{cases}$



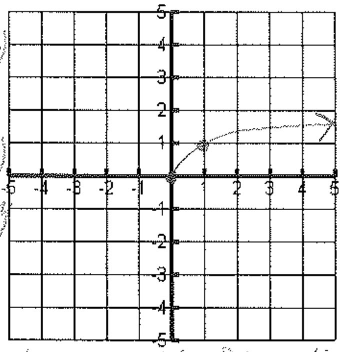
Domain: $(-\infty,\infty)$
Continuity Interval $(-\infty,\infty)$
Even Symmetry
$\lim_{x\to-\infty} y = \infty$
$\lim_{x\to\infty} y = \infty$
no discontinuities

Rational: $y = \frac{1}{x}$



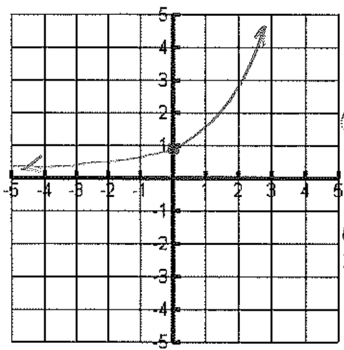
Domain: $(-\infty,0)\cup(0,\infty)$
Continuity Interval $(-\infty,0)\cup(0,\infty)$
Odd Symmetry
$\lim_{x\to-\infty} y = 0$
$\lim_{x\to\infty} y = 0$
Discontinuities: x=0 infinite

Root: $y = \sqrt{x}$



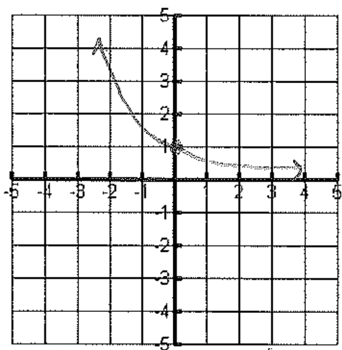
Domain: $[0,\infty)$
Continuity Interval $(0,\infty)$
No Symmetry
$\lim_{x\to-\infty} y = DNE$
$\lim_{x\to\infty} y = \infty$
No Discontinuities

Exponential Growth: $y = e^x$ where $x > 0$



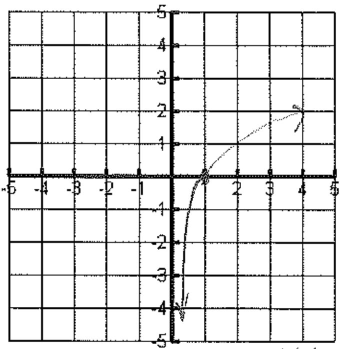
Domain: $(-\infty,\infty)$
Interval of Continuity: $(-\infty,\infty)$
No Symmetry
$\lim_{x\to-\infty} y = 0$
$\lim_{x\to\infty} y = \infty$
No discontinuities

Exponential Decay: $y = e^x$ where $x < 0$



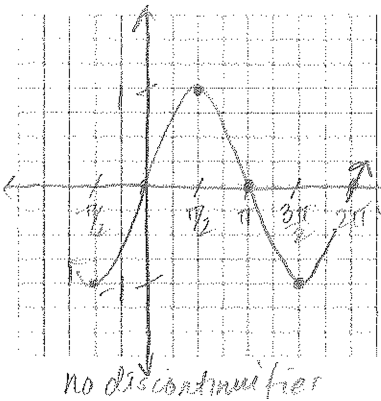
Domain: $(-\infty,\infty)$
Interval of Continuity: $(-\infty,\infty)$
No Symmetry
$\lim_{x\to-\infty} y = \infty$
$\lim_{x\to\infty} y = 0$
No discontinuities

Logarithmic Growth: $y = \ln x$



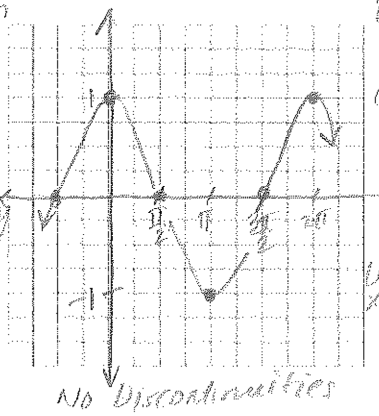
Domain $(0,\infty)$
Continuity interval: $(0,\infty)$
No symmetry
$\lim_{x\to-\infty} y = DNE$
$\lim_{x\to\infty} y = \infty$
No discontinuities

Sine: $y = \sin(x)$ Cosine: $y = \cos(x)$ Tangent: $y = \tan(x)$



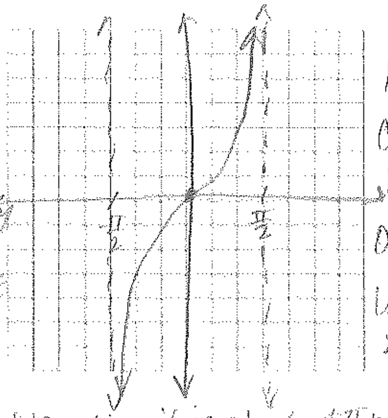
Domain $(-\infty, \infty)$
Cont. $(-\infty, \infty)$
Odd Symmetry
$\lim_{x \to \pm\infty} y =$ DNE

no discontinuities



Domain $(-\infty, \infty)$
Cont. $(-\infty, \infty)$
Even Symmetry
$\lim_{x \to \pm\infty} y =$ DNE

No Discontinuities



Domain: $\mathbb{R}, \neq \pm\frac{\pi}{2}n$
Continuity $\mathbb{R}, \neq \pm\frac{\pi}{2}n$
Odd Symmetry
$\lim_{x \to \pm\infty} y =$ DNE

Discontinuities at $x = \pm\frac{\pi}{2}n$

Analyze the following functions without graphing them on a graphing calculator (except where noted).

9) $f(x) = x^2 - 2x - 3 = (x-3)(x+1) = (x-1)^2 - 4$

Domain: $(-\infty, \infty)$ polynomial

Range: $[-4, \infty)$

x-intercept(s):

$0 = x^2 - 2x - 3$ $x = 3, -1$
$0 = (x-3)(x+1)$

y-intercept(s): $y = 0 - 0 - 3$
$= -3$

Extrema (by hand):

$x = \frac{-b}{2a} = \frac{2}{2} = 1$ or $x^2 - 2x - 3 =$
$(1, -4)$ absolute min. $(x-1)^2 - 3 - 1 =$
$(x-1)^2 - 4$
$(1, 4)$ absolute min.

End Behavior:

$\lim_{x \to -\infty} f(x) = \infty$

$\lim_{x \to \infty} f(x) = \infty$

Interval of Continuity: $(-\infty, \infty)$

Tests for Symmetry:

$f(-x) = (-x)^2 - 2(-x) - 3$
$= x^2 + 2x - 3$
none

10) $g(x) = 6(x+2)^2 - 4(x+2)^3 = 2(x+2)^2[3 - 2(x+2)]$
$2(x+2)^2(-2x-1)$
$= -2(x+2)^2(2x+1)$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

odd degree
negative leading coeff. ∴

x-intercept(s):

$x = -2, -\frac{1}{2}$

y-intercept(s): $y = 6(2)^2 - 4(2)^3 = 24 - 32 = -8$

End Behavior:

$\lim_{x \to -\infty} g(x) = \infty$

$\lim_{x \to \infty} g(x) = -\infty$

Interval of Continuity: $(-\infty, \infty)$

**Rational Expressions and Functions:**
Domain and Range
End Behavior (limits at infinity): $\lim_{x\to\pm\infty} f(x) = horizontal\ asymptote$
Discontinuity: infinite discontinuities occur at vertical Asymptotes and removable discontinuities occur at holes.

11) Determine the following for the function $f(x) = \frac{3x^2+5x+2}{2x^2+x-1} = \frac{(3x+2)\cancel{(x+1)}}{(2x-1)\cancel{(x+1)}}$

Domain: $(-\infty,-1) \cup (-1, 1/2) \cup (1/2, \infty)$

Range: $(-\infty, 3/2) \cup (3/2, \infty)$

x-intercept(s): $x = -2/3$

y-intercept(s): $y = -2$

End Behavior:

$\lim_{x\to-\infty} f(x) = 3/2$

$\lim_{x\to\infty} f(x) = 3/2$

Interval of Continuity:

$(-\infty,-1) \cup (-1, 1/2) \cup (1/2, \infty)$

Name & Location of Disc: $x=-1$ removable
$x = 1/2$ infinite

12) Simplify

a. $\frac{3e^x x^2-3e^x}{e^{4x}(x+1)^3} = \frac{3e^x(x^2-1)}{e^{4x}(x+1)^3}$

$= \frac{3\cancel{(x+1)}(x-1)}{e^{3x}(x+1)^{\cancel{3}2}} = \frac{3(x-1)}{e^{3x}(x+1)^2}$

b. $\frac{3x^{-2}}{1x^{-1}-\frac{1}{3}x^{-4}} = \frac{\left(\frac{3}{x^2}\right)}{\left(\frac{1}{x}-\frac{1}{3x^4}\right)} \left(\frac{3x^4}{3x^4}\right)$

$= \frac{9x^2}{3x^3-1}$

c. $\frac{\left(\frac{1}{x+1}\right)}{\left(\frac{2}{x-3}-\frac{1}{x^2-2x-3}\right)} \frac{(x-3)(x+1)}{(x-3)(x+1)}$

$x^2-2x-3 = (x-3)(x+1)$

$\frac{x-3}{2(x+1)-1} = \frac{x-3}{2x+1}$

13) Solve

a. $\frac{x+3}{x^2-2x-8} - \frac{x-5}{x^2-12x+32} \le 0$

$(x-4)(x+2) \quad (x-4)(x-8)$

$\frac{(x+3)(x-8)-(x-5)(x+2)}{(x-4)(x+2)(x-8)} \le 0$

$\frac{\cancel{x^2}-5x-24-\cancel{x^2}+3x+10}{(x-4)(x+2)(x-8)} \le 0$

$\frac{-2x-14}{(x-4)(x+2)(x-8)} \le 0$

$\frac{-2(x+7)}{(x-4)(x+2)(x-8)} \le 0$

CV $x = -7, 4, -2, 8$

zero, und, und, und

$- \quad + \quad - \quad + \quad -$

$-7 \quad -2 \quad 4 \quad 8$

$(-\infty, -7] \cup (-2, 4)$
$\cup (8, \infty)$

b. $\frac{5x(x^2+1)\,(x+2)^{-1}+10\qquad)(x^2+1)}{20(x+2)(x^2+1)^3} \ge 0$

$\frac{\frac{5x(x^2+1)}{x+2}+10\ (x^2+1)}{20(x+2)(x^2+1)^3} \cdot \frac{(x+2)}{(x+2)}$

$\frac{5x(x^2+1)+10(x+2)(x^2+1)}{20(x+2)^2(x^2+1)^3}$

$\frac{\cancel{5}(x^2+1)[x+2(x+2)]}{4\,\cancel{20}(x+2)^2(x^2+1)^{\cancel{3}2}}$

$\frac{3x+4}{4(x+2)^2(x^2+1)^2}$

CV: $x = -4/3, -2$

$- \quad - \quad +$

$-2 \quad -4/3$

$[-4/3, \infty)$

## FUNCTION OPERATIONS AND COMPOSITION; INVERSE FUNCTIONS

**Composition** of a function $g$ with a function $f$ is defined as:

$$h(x) = g(f(x))$$

The domain of h is the set of all x-values such that x is in the domain of f and f(x) is in the domain of g.

**Inverses:** Functions f and g are inverses of each other provided:

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x$$

The function g is denoted as $f^{-1}$, read as "f inverse".

**Horizontal Line Test:** If any horizontal line which is drawn through the graph of a function f intersects the graph no more than once, then f is said to be a **one-to-one** function and has an inverse.

14) Let f and g be functions whose values are given by the table below. Assume g is one-to-one with inverse $g^{-1}$.

a. $f(g(3)) = f(4) = -1$

b. $g^{-1}(4) = 3 \rightarrow g(x) = (3,4)$ so $g^{-1}(x) = (4,3)$

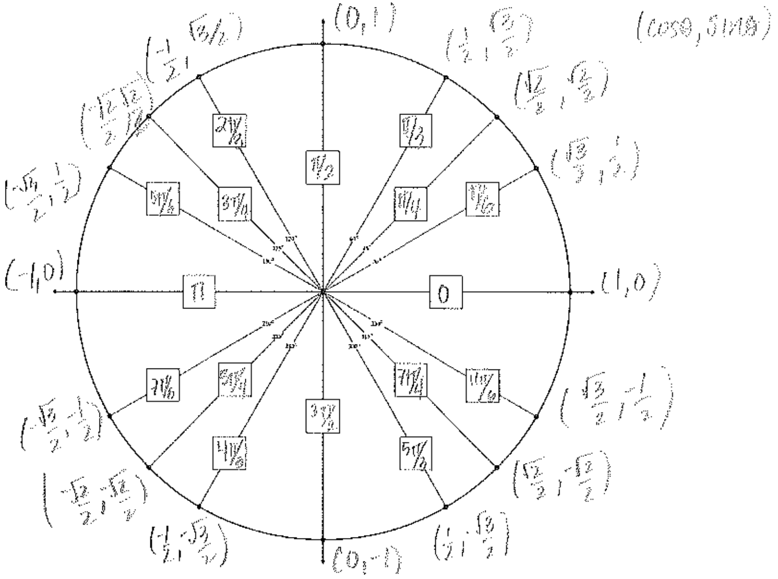
c. $f(g^{-1}(6)) = f(4) = -1$

d. $f^{-1}(f(g(2))) = f^{-1}(f(3)) = f^{-1}(10) = 3$

| X | f(x) | g(x) |
|---|---|---|
| 1 | 6 | 2 |
| 2 | 9 | 3 |
| 3 | 10 | 4 |
| 4 | -1 | 6 |

## TRIGONOMETRY

UNIT CIRCLE: Complete angles and Sine/Cosine



Key Trig Identities – formulas you should know:

$$\sin^2 x + \cos^2 x = 1$$
$$\sin(2x) = 2\sin x \cos x$$
$$\cos(2x) = 1 - 2\sin^2 x$$
$$1 + \tan^2 x = \sec^2 x$$
$$\sin(-x) = -\sin x \quad ODD$$
$$\cos(-x) = \cos x \quad EVEN$$

15) Without using a calculator or table, find each value:

a. $\cos\left(-\frac{\pi}{3}\right)$ $\frac{1}{2}$

b. $\sec\left(\frac{11\pi}{6}\right)$ $\frac{2}{\sqrt{3}}$ or $\frac{2\sqrt{3}}{3}$

c. $\tan^{-1}(-1)$ $\frac{-\pi}{4}$

d. $\sec^{-1}(-2)$ $\frac{2\pi}{3}$

16) Solve the trigonometric equations algebraically by using identities and **without** the use of a calculator. Find all solutions in the interval $0 \le \theta \le 2\pi$.

a. $2\sin^2\theta - 1 = 0$

$2\sin^2\theta = 1$

$\sin^2\theta = 1/2$

$\sin\theta = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

b. $\sin(2x) = \cos x$

$2\sin x\cos x = \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}, \pi/6, \frac{5\pi}{6}$

c. $2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

$\theta = \frac{\pi}{3}, 5\pi/3, \pi$

## EXPONENTIAL AND LOGARITHMIC FUNCTIONS:

**Logarithm:** For any positive numbers b and y with $b \neq 1$, we define the **logarithm of y with base b** as follows: $\log_b y = x$ if and only if $b^x = y$

**LAWS OF LOGARITHMS** (for M, N, $b > 0,\ b \neq 1$)

$(i)\ \log_b MN = \log_b M + \log_b N$

$(ii)\ \log_b \frac{M}{N} = \log_b M - \log_b N$

$(iii)\ \log_b M = \log_b N$ *if and only if* $M = N$

$(iv)\ \log_b M^k = k \cdot \log_b M$

$(v)\ b^{\log_b M} = M$

The logarithmic and exponential functions are **inverse** functions.

Example: Consider $f(x) = 2^x$ and $g(x) = \log_2 x$. Verify that (3,8) is on the graph of f and (8,3) on the graph of g. In addition, $f(g(x)) = 2^{\log_2 x} = x$ *and* $g(f(x)) = \log_2(2^x) = x$ which verifies that f and g are indeed inverse functions.

Simplify.

17) $\log_5 \frac{1}{125} = \log_5 5^{-3}$

$= -3$

18) $\log_4 \sqrt[8]{16} = \log_4 (4^2)^{1/8}$

$\frac{2}{8} = \frac{1}{4}$

19) $2\ln\frac{3}{e^3}$

$2\ln 3 - 2\ln e^3$

$2\ln 3 - 2(3) = 2\ln 3 - 6$

20) $\frac{1}{t}\ln e^t$

$\frac{1}{t}(t) = 1$

21) $e^{3\ln(x+5)}$

$e^{\ln(x+5)^3}$

$= (x+5)^3$

22) $e^{-lnx}$

$= e^{\ln x^{-1}}$

$= x^{-1}$

$= \frac{1}{x}$

23) $4\ln e^{x^2}$

$4x^2$

24) $e^{3x+ln5}$

$= e^{3x} \cdot e^{\ln 5}$

$= 5e^{3x}$

25) $e^{3(\ln(x)+\ln 5)}$

$e^{3\ln x + 3\ln 5}$

$= e^{\ln x^3} \cdot e^{\ln 5^3}$

$= x^3 \cdot 5^3 = 125x^3$

26) Write $5\,ln\,(x) - 3\,ln\,(y) + 2\,ln\,(4x) - 6\,ln\,(y^2)$ as a a single logarithm.

$\ln x^5 - \ln y^3 + \ln 4^2x^2 - \ln y^{12} = \ln\left(\frac{x^5 \cdot 16 \cdot x^2}{y^3 \cdot y^{12}}\right) = \ln\left(\frac{16x^7}{y^{15}}\right)$

27) Expand $\ln\left(\frac{3x^2}{2\sqrt{y^2-4}}\right)$. $= \ln 3 + 2\ln x - \ln 2 - \frac{1}{2}\ln(y^2-4)$

28) Evaluate:

a. $\ln 1$

$0$

b. $\ln 3e$

$\ln 3 + \ln e$

$= 1 + \ln 3$

c. $\ln e$

$1$

d. $\ln 0$

DNE

e. $\ln(\ln e)$

$\ln(1)$

$0$

Limits: $\lim_{x \to c} f(x) = N$ "Limit as x approaches c of f(x) equals N"

## LIMIT ≠ CONTINUITY

Understanding the difference between limits and continuity: Limit is the value (y) that the function APPROACHES as you get close to c (either from one side or from both sides). Continuity implies that the limit exists (from both sides) and that the limit = the value at c!

**Limit Exists:** $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) \;\therefore\; \lim_{x \to c} f(x)\ exists$ and will equal all the same value or ±∞.

**Continuous:** $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)\ AND = f(c) = A\ (where\ A\ is\ a\ constant\ not\ \pm\infty)$ **MUST SHOW ALL 3!!**

**One-Sided limit**: exists if graph approaches a value from one side: + (right) or – (left)

**Evaluating Limits:**

1. **Tables:** either from table or by graphing and viewing table looking for values of y approaching the same number. Check one sided limits first and if they are equal the overall limit exists.
2. **Graphically:** Check one sided limits (again – value of limit is the y-value or the height of the graph) and if they are equal overall limit exists.
3. **Algebraically – Direct substitution:** plug in c. If you get a constant then limit exists. Rational functions may need to be simplified first!

   When you use Direct substitution and you get

   - $\frac{0}{number}$ then the limit is equal to 0
   - $\frac{0}{0}$ most likely a removable discontinuity, try simplifying or rationalizing.
   - $\frac{number}{0}$ most likely an infinite discontinuity, look at each side and evaluate one sided limit or if asking for two sided limit make sure the function is approaching the same from both sides.

$\lim_{x \to \pm\infty} f(x)$: **End Behavior:**

- Polynomial: use degree (even or odd) and sign of leading coefficient to decide parabolic or cubic.
- Rational: end behavior is determined by the Horizontal or Oblique Asymptotes.
- Other "Known" Graphs: know

Examples: Evaluate the limits:

29) $\lim\limits_{x\to 3}(2x^2)$

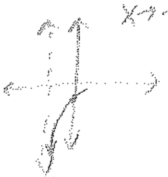
$= 2(3)^2$

$= 2\cdot 9$

$= 18$

30) $\lim\limits_{x\to -2^+}\frac{1}{x^2-4} =$

$\lim\limits_{x\to -2^+}\frac{1}{(x+2)(x-2)} = \frac{1}{0}$ VA



$= -\infty$

31) $\lim\limits_{x\to \frac{\pi}{2}}(x\sin x)$

$= \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)$

$= \pi/2\,(1) = \frac{\pi}{2}$

32) $\lim\limits_{x\to 5}\frac{3}{x-5} = \frac{3}{0}$ VA

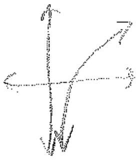
DNE b/c

$\lim\limits_{x\to 5^-}\frac{3}{x-5} = -\infty$

$\neq$

$\lim\limits_{x\to 5^+}\frac{3}{x-5} = \infty$

33) $\lim\limits_{x\to 0^+}\ln(x)$ "Known graph"



$= -\infty$

34) $\lim\limits_{x\to -\frac{\pi}{2}}\tan(x)$ "Known graph"

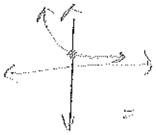
DNE b/c

$\lim\limits_{x\to -\frac{\pi}{2}^-}\tan x = \infty$

$\neq$

$\lim\limits_{x\to -\frac{\pi}{2}^+}\tan x = -\infty$

35) $\lim\limits_{x\to -\infty} e^{-x}$ "Known graph"



$= \infty$

36) $\lim\limits_{x\to 1}\frac{3x^2-2x-1}{x^2-1} = \frac{0}{0}$ hole

$= \lim\limits_{x\to 1}\frac{(3x+1)(x-1)}{(x+1)(x-1)}$

$= \frac{3(1)+1}{1+1} = \frac{4}{2} = 2$

37) If $f(x) = \begin{cases} 2x-3, & if\ x < -1 \\ 2, & if\ x = -1 \\ 5x, & if\ x > -1 \end{cases}$, determine the following:

a. $\lim\limits_{x\to -1^-} f(x)$

$= 2(-1)-3 = -5$

b. $\lim\limits_{x\to -1^+} f(x)$

$5(-1) = -5$

c. $\lim\limits_{x\to -1} f(x)$

$= -5$

d. Is f continuous?

No, b/c $\lim\limits_{x\to -1} f(x) = -5$

$\neq f(-1) = 2$

38) If $f(x) = |x+2| = \begin{cases} x+2, & x \ge -2 \\ -(x+2), & x < -2 \end{cases}$, determine the following:

a. $\lim\limits_{x\to -2^-} f(x)$

$-(-2+2) = 0$

b. $\lim\limits_{x\to -2^+} f(x)$

$-2+2 = 0$

c. $\lim\limits_{x\to -2} f(x)$

$= 0$

d. Is f continuous?

yes because

$\lim\limits_{x\to -2} f(x) = 0 = f(-2)$

39) Determine whether $f(x)$ is continuous. Justify. Then, determine domain, interval of continuity and location and type of any discontinuities.

a. $f(x) = \begin{cases} 3x^2, & x < 0 \\ 4, & x = 0 \\ 3\sin^2 x, & x > 0 \end{cases}$ $x=0$ removable disc.

$\lim\limits_{x\to 0^-} f(x) = 3(0^2) = 0$

$f(0) = 4$

$\lim\limits_{x\to 0^+} f(x) = 3(\sin 0)^2 = 0$

Domain: $(-\infty,\infty)$ Interval: $(-\infty,0)\cup(0,\infty)$

b. $f(x) = \begin{cases} \ln x, & x < e^2 \\ \sqrt{x}, & x \ge e^2 \end{cases}$ $x = e^2$ jump discont.

$\lim\limits_{x\to e^{2-}} f(x) = \ln e^2 = 2$

$\neq$

$\lim\limits_{x\to e^{2+}} f(x) = f(e^2) = \sqrt{e^2} = e$

Domain: $(-\infty,\infty)$ Interval: $(-\infty,e^2)\cup(e^2,\infty)$

c. $f(x) = \begin{cases} 3x^2-1, & x \le 2 \\ 5x+1, & x > 2 \end{cases}$

$\lim\limits_{x\to 2^-} f(x) = f(2) = 3(2)^2-1 = 11$

$=$ yes!

$\lim\limits_{x\to 2^+} f(x) = 5(2)+1 = 11$

Domain = Int. of cont $= (-\infty,\infty)$

40) If $\lim\limits_{x\to c} f(x) = -2$ and $\lim\limits_{x\to c} g(x) = 5$, determine the following:

a. $\lim\limits_{x\to c} 5f(x)$

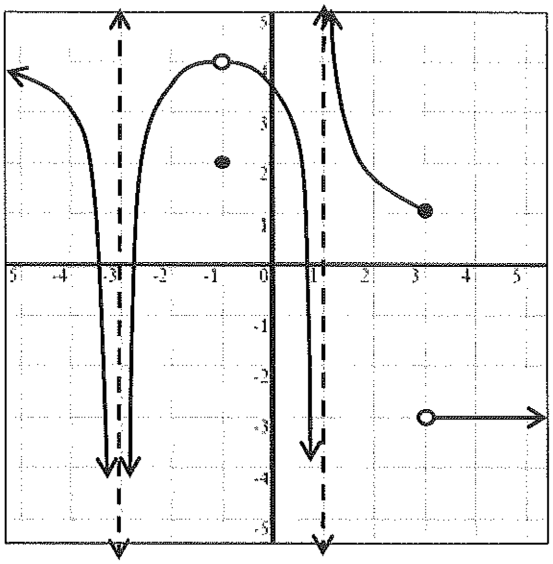
$5(-2) = -10$

b. $\lim\limits_{x\to c}[f(x) - 3g(x)]$

$-2 - 3(5) = -17$

c. $\lim\limits_{x\to c}\frac{(f(x))^3}{\sqrt[3]{g(x)}} = \frac{(-2)^3}{\sqrt[3]{5}} = \frac{-8}{\sqrt[3]{5}}$ OR $\frac{-8\sqrt[3]{5^2}}{5}$

41) Let f(x) be the graph below. Determine the following:



a. $\lim_{x\to -3^-} f(x) = -\infty$

b. $\lim_{x\to -3^+} f(x) = -\infty$

c. $\lim_{x\to -3} f(x) = -\infty$

d. f(-3) = und.

e. Is f continuous at -3? no b/c f(-3) is undefined

f. $\lim_{x\to -1^-} f(x) = 4$

g. $\lim_{x\to -1^+} f(x) = 4$

h. $\lim_{x\to -1} f(x) = 4$

i. f(-1) = 2

j. $\lim_{x\to 1^-} f(x) = -\infty$

k. $\lim_{x\to 1^+} f(x) = \infty$

l. $\lim_{x\to 1} f(x) =$ DNE

m. $f(1) =$ und

n. $\lim_{x\to 3^-} f(x) = 1$

o. $\lim_{x\to 3^+} f(x) = -3$

p. $\lim_{x\to 3} f(x) =$ DNE

q. $f(3) = 1$

r. $\lim_{x\to -\infty} f(x) = \infty$

s. $\lim_{x\to \infty} f(x) = -3$

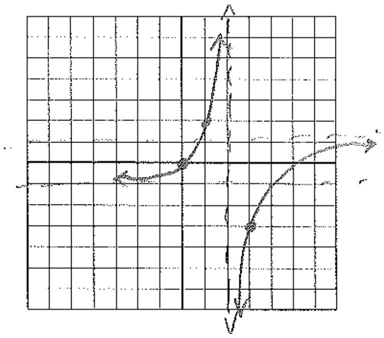
t. Continuity Interval: $(-\infty,-3)\cup(-3,-1)\cup(-1,1)\cup(1,3)\cup(3,\infty)$

u. Name and type of each discontinuity:
x = -3 infinite
x = -1 removable pt.
x = 1 infinite
x = 3 jump

v. Domain:
$(-\infty,-3)\cup(-3,1)\cup(1,\infty)$

42) Draw a graph given the following conditions:

- ♦ $f(0) = 0$
- ♦ $f(1) = 2$
- ♦ $f(4) = -3$
- ♦ $\lim_{x\to 3^-} f(x) = \infty$
- ♦ $\lim_{x\to -\infty} f(x) = -1$
- ♦ $\lim_{x\to \infty} f(x) = 1$
- ♦ $\lim_{x\to 3^+} f(x) = -\infty$



43) Draw a graph given the following conditions:

- ♦ $f(-2) = 0$
- ♦ $f(2) = 2$
- ♦ $\lim_{x\to -2^-} f(x) = 4$
- ♦ $\lim_{x\to -\infty} f(x) = -\infty$
- ♦ $\lim_{x\to \infty} f(x) = 5$
- ♦ $\lim_{x\to -2^+} f(x) = 0$