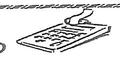
ALGEBRA TOPICS

LAWS OF EXPONENTS

- What you need to know:

Assume a and b are real numbers and m and n positive integers.
(i) $a^m \cdot a^n = a^{m+n}$ (ii) $(ab)^m = a^m \cdot b^m$ (iii) $(a^m)^n = a^{mn}$
(iv) $a^{-m} = \frac{1}{a^m}$ (v) $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

PRACTICE PROBLEMS (NO  , please!)

Simplify:

1) $(2c^2d^3)^3$ $2^3 c^6 d^9$

2) $(a^2b)(-3ab^3)(-2ab)^{-2}$ $\frac{-3ab^2}{4}$

3) $r^{h-2}(r^{h+1})^2 = r^{h-2} \cdot r^{2h+2} = r^{3h}$

4) Solve for n: $3^{5n} = 3^5(3^{2n})^2$
$3^{5n} = 3^{5+4n}$
$5n = 5+4n$
$\boxed{n=5}$

5) $4^{n+3} \cdot 16^n = 8^{3n}$
$2^{2n+6} \cdot 2^{4n} = 2^{9n} \Rightarrow 2^{6n+6} = 2^{9n} \Rightarrow \boxed{n=2}$

6) Write in exponential form (no negative exponents): $\sqrt[3]{8x^6y^{-4}}$ $\left(\frac{8x^6}{y^4}\right)^{\frac{1}{3}}$

7) Write in simplest radical form: $\sqrt[4]{27} \cdot \sqrt[8]{9}$
$3^{3/4} \cdot 3^{1/4} = 3$

8) Simplify. Give your answer in exponential form: $a^{\frac{1}{2}}\left(a^{\frac{3}{2}} - 2a^{\frac{1}{2}}\right)$
$a^2 - 2a$

9) Using the property of exponents, show that $27^{\frac{4}{3}} - 9^{\frac{3}{2}} = 2 \cdot 3^3$ (without the help of a  , of course!)
$27^{\frac{4}{3}} - 9^{\frac{3}{2}} = (3^3)^{\frac{4}{3}} - (3^2)^{\frac{3}{2}} = 3^4 - 3^3 = 3^3(3-1) = 3^3 \cdot 2$
↓
Factor out 3^3 !

Solve (algebraic solution!):

10) $(3x+1)^{\frac{3}{4}} = 8$
$3x+1 = 8^{\frac{4}{3}}$
$3x+1 = 2^4$
$\boxed{x=5}$

11) $9x^{-\frac{2}{3}} = 4$
$x^{\frac{2}{3}} = \frac{4}{9} \Rightarrow x = \left(\frac{4}{9}\right)^{\frac{3}{2}} \Rightarrow x = \left(\frac{2^2}{3^2}\right)^{\frac{3}{2}}$
$\boxed{x = \frac{8}{27}}$

FACTORING POLYNOMIALS

What you need to know:

- Perfect Square Trinomials: $a^2 + 2ab + b^2 = (a+b)^2$
$a^2 - 2ab + b^2 = (a-b)^2$
- Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$
- Sum of Cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- Difference of Cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

PRACTICE PROBLEMS – Factor completely:

1) $3x^2 - x - 10$ $(3x+5)(x-2)$

2) $4x^2 + 12x - 7$ $(2x-1)(2x+7)$

3) $a^2 - 10a + 25$ $(a-5)^2$

4) $4a^2 - 4ab + b^2$ $(2a-b)^2$

5) $25x^2 - 16y^2$ $(5x-4y)(5x+4y)$

6) $3x^5 - 48x = 3x(x^4-16) = 3x(x^2+4)(x^2-4) =$
$= 3x(x^2+4)(x+2)(x-2)$

7) $x^6 - y^6 = (x^2-y^2)(x^4+x^2y^2+y^4)$
$= (x+y)(x-y)(x^4+x^2y^2+y^4)$

8) $64 - z^6 = (2^2-z^2)(2^4+4z^2+z^4) =$
$= (2+z)(2-z)(16+4z^2+z^4)$

9) $8p^3 + 1$
$(2p)^3+1 = (2p+1)(4p^2-2p+1)$

10) $x(y-2) + 3(2-y)$
$x(y-2) - 3(y-2) = (y-2)(x-3)$

11) $x^2 - 6x + 9 - 4y^2$
$(x-3)^2 - 4y^2 = (x-3+2y)(x-3-2y)$

12) $(x+y)^3 + (x-y)^3$ Let $x+y=a$, $x-y=b$
use sum of cubes formula above to get
$= 2x\,[3y^2 + x^2]$

13) $6x^2 - 7xy - 3y^2 =$
$(3x+y)(2x-3y)$

14) $4x^3 + 8x^2y - 5xy^2$
$= x(4x^2 + 8xy - 5y^2)$
$= x(2x-y)(2x+5y)$

SOLVING POLYNOMIAL INEQUALITIES

Example 1: $x^2 - 4x - 5 < 0$

Solution: $(x+1)(x-5) < 0$ *factor*

mark the zeros; pick a test point to determine the sign of the polynomial in each interval

The solution set of this conjunction is $\{x : -1 < x < 5\}$.

Example 2: $-2x^2 + 8x + 10 < 0$

$-2(x+1)(x-5) < 0$ Factor

Make a sign chart; mark the zeros; pick a test point in each interval and plug it into the factored form to determine the sign of each interval

The solution set of this disjunction is: $\{x : x \le -1 \text{ or } x \ge 5\}$

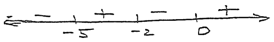
PRACTICE PROBLEMS

1) $x^3 + 7x^2 + 10x > 0$

$x(x^2+7x+10) > 0$

$x(x+5)(x+2) > 0$

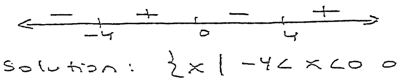
sign chart:



Solution: $\{x \mid -5 < x < -2 \text{ or } x > 0\}$

2) $x^3 - 16x > 0$

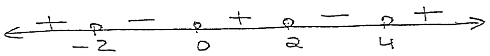
$x(x+4)(x-4) > 0$



Solution: $\{x \mid -4 < x < 0 \text{ or } x > 4\}$

3) $(x^2 - 4x)(x^2 - 4) \le 0$

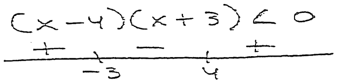
$x(x-4)(x+2)(x-2) < 0$



$\{x \mid -2 \le x \le 0 \text{ or } 2 \le x \le 4\}$

4) $x^2 - x - 12 \overset{<}{\cancel{\ge}} 0$

$(x-4)(x+3) < 0$



$\{x \mid -3 < x < 4\}$

PRE-CALCULUS TOPICS

FUNCTIONS

You are given a polynomial equation and one or more of its roots. Find the remaining roots.

What you need to know:

(i) **The Remainder Theorem:** When a polynomial P(x) is divided by (x-a), the remainder is P(a);

(ii) **Factor Theorem:** For a polynomial P(x), (x-a) is a factor if and only if P(a)=0;

(iii) Use synthetic division when dividing a polynomial by a linear factor; long division will always work;

(iv) **Odd Function:** A function f(x) is odd ⇔ f(-x) = -f(x);

(v) **Even Function:** A function f(x) is even ⇔ f(-x) = f(x);

Remarks: item (iv) means the graph of the function has rotational symmetry about the origin, and (v) means the graph has reflectional symmetry about the y-axis;

Puzzled? Let's try some...

Find the quotient and the remainder when the first polynomial is divided by the second. No calculators allowed!

1) $x^3 - 2x^2 + 5x + 1;\ x - 1$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & 5 & 1 \\ & & 1 & -1 & 4 \\ \hline & 1 & -1 & 4 & 5 \end{array}$$

Quotient: $x^2 - x + 4$

Remainder: 5

2) $3x^4 - 2x^3 + 5x^2 + x + 1;\ x^2 + 2x$

→ Q: $3x^2 - 8x + 21$

R: $-41x + 1$

$$\begin{array}{r l} & 3x^2 - 8x + 21 \\ \hline x^2+2x \,) & 3x^4 - 2x^3 + 5x^2 + x + 1 \\ & -(3x^4 + 6x^3) \\ \hline & -8x^3 + 5x^2 + x + 1 \\ & -(-8x^3 - 16x^2) \\ \hline & 21x^2 + x + 1 \\ & -(21x^2 + 42x) \\ \hline & -41x + 1 \end{array}$$

3) Determine whether x-1 or x+1 are factors of $x^{100} - 4x^{99} + 3$.

$(x-1)$ is a Factor $\Leftrightarrow P(1) = 0$

$P(1) = 1 - 4 + 3 = 0 \Rightarrow (x-1)$ is a Factor.

$(x+1)$ is a Factor $\Leftrightarrow P(-1) = 0$

$P(-1) = 1 + 4 + 3 = 8 \ne 0 \Rightarrow (x+1)$ is not A FACTOR.

4) When a polynomial P(x) is divided by $3x-4$, the quotient is x^3+2x+2 and the remainder is -1. Find P(x). (recall: P(x) = (divisor)x(quotient) + remainder!)

$P(x) = (x^3+2x+2)(3x-4) - 1$

$P(x) = 3x^4 - 4x^3 + 6x^2 - 8x + 6x - 8 - 1 \Rightarrow \boxed{P(x) = 3x^4 - 4x^3 + 6x^2 - 2x - 9}$

5) You are given a polynomial equation and one or more of its roots. Find the remaining roots. $2x^4-9x^3+2x^2+9x-4=0$; roots: $x = -1$, $x = 1$.

Since $(x-1)$ and $(x+1)$ are factors $\Rightarrow (x+1)(x-1) = x^2-1$ is a factor

Divide to find remaining roots:

$$x^2-1 \overline{\smash{)}\,2x^4-9x^3+2x^2+9x-4}\quad\text{quotient } 2x^2-9x+4$$

$\Rightarrow 2x^2-9x+4 = (2x-1)(x-4)$ are 2 other factors

Remaining roots: $\boxed{x = 1/2 \text{ and } x = 4}$

Verify algebraically whether the functions are odd or even.

6) $f(x) = 5x^3 - x$

$f(-x) = 5(-x)^3 - (-x)$

$= -5x^2 + x$

$= -f(x)$ ∴ ODD!

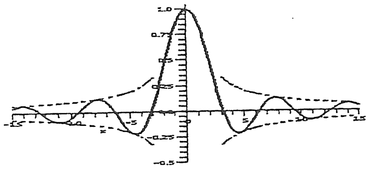
7) $h(x) = \dfrac{5}{x^2+1}$

$h(-x) = \dfrac{5}{(-x)^2+1} = \dfrac{5}{x^2+1}$ ∴ even!

FINDING THE ASYMPTOTES OF A RATIONAL FUNCTION

What you need to know:

- **Vertical Asymptote:** Set denominator = 0 and solve it. For values of x near the asymptote, the y-values of the function "approach" infinity (i.e., the y-values get increasingly large...)
- **Horizontal or Oblique Asymptote:** when the values of x get very large (approach infinity), the y-values tend to the asymptote. Below is an example of a curve displaying asymptotic behavior:



A curve can intersect its asymptote, even infinitely many times. This does not apply to vertical asymptotes of functions.

- To find the horizontal or oblique asymptote, divide the numerator by the denominator. The asymptote is given by the quotient function.

PRACTICE PROBLEMS (No 🖩, please!)

1) For the functions below determine: a) the domain; b) the x-and y-intercepts; c) the vertical, horizontal or oblique asymptotes; and d) whether R(x) is even or odd (show algebraic analysis).

1) $R(x) = \dfrac{3x^2-3x}{x^2+x-12}$

a) $\{x \in \mathbb{R} \setminus x \neq 3, x \neq -4\}$

b) x-intercept: $x = 0, 1$

y-intercept: $(0, 0)$

c) V.A: $x = 3$; $x = -4$

H.A: $y = 3$

d) neither even nor odd.

2) $r(x) = \dfrac{3x^2+5x-2}{x^2-4}$. (explain what happens at x=-2...)

a) $\{x \in \mathbb{R} \mid x \neq \pm 2\}$

b) x-intercept: $(\frac{1}{3}, 0)$

y-intercept: $(0, \frac{1}{2})$

c) V.A: $x = 2$; there is a "hole" at $x = -2$

H.A: $y = 3$

d) neither

3) Consider $f(x) = \dfrac{8}{2+x^2}$. a) is f even or odd? b) explain why there are no vertical asymptotes; c) What is the domain of f? Range? d) Given the equation of its horizontal asymptote.

a) $f(-x) = \dfrac{8}{2+(-x)^2} = \dfrac{8}{2+x^2} = f(x) \Rightarrow$ even!

b) Since the denominator $2 + x^2 \neq 0$ for every real numb

There are no V.A's.

c) Domain: $\{x \in \mathbb{R}\}$

Range: $\{y \in \mathbb{R} \mid 0 < y \leq 4\}$ (The max y-value is attained when $x = 0$)

d) $y = 0$

4) Find a quadratic function f(x) with y-intercept -2 and x-intercepts (-2,0) and $\left(\frac{1}{3},0\right)$. $f(x) = 3x^2+5x-2$

Set up: $f(x) = a(x+2)(x-\frac{1}{3})$; when $x=0$, $y=-2$ $\Rightarrow$ $-2 = a(2)(-\frac{1}{3})$ $\Rightarrow$ $a=3$ etc...

RATIONAL ALGEBRAIC EXPRESSIONS AND FUNCTIONS

What you need to know:

(i) A rational function is a function of the form $\frac{p(x)}{q(x)}$, where p and q are polynomial functions and $q(x) \neq 0$ for every x.

(ii) To simplify a rational expression, factor both numerator and denominator. Cancel out common factors.

(iii) To find the domain of a rational function: solve q(x) = 0. The zeroes of the polynomial q(x) must be excluded from the domain.

(iv) To find the zeroes of a rational function: set p(x) = 0 and solve it. The zeros of p(x) are the zeros of the polynomial function (provided they are not also zeros of the denominator q(x)!)

In #1-3: Find (a) the domain of each function; and (b) the zeros, if any. No calculators, please!

1) $g(x)=\frac{2x^2+3x-9}{x^3-4x} = \frac{(2x-3)(x+3)}{x(x+2)(x-2)}$ Domain $\{x \mid x \neq 0, -2, 2\}$

For domain, set denominator = 0 :

For zeros or roots, set numerator = 0 (provided no common factors with denominator exist!) Zeros : $x = 3/2, -3$

2) $g(x)=\frac{(x+3)(x-2)(x-1)}{x^2-6x+8} = \frac{(x+3)(x-2)(x-1)}{(x-2)(x-4)}$

Domain : $\{x \in \mathbb{R} \mid x \neq 2, 4\}$

Zeros : $x = -3, 1$ only

the function has a "hole" at $x = 2$!

3) $h(x)=\frac{x^3+x^2-x-1}{x^3-x^2-x+1} = \frac{x^2(x+1)-(x+1)}{x^2(x-1)-(x-1)} = \frac{(x+1)(x^2-1)}{(x-1)(x^2-1)} = \frac{x+1}{x-1}$

Domain . $\{x \mid x \neq -1, 1\}$

Zeros : No zeros ! (The only candidate for a zero would be $x=-1$ which is not in the domain of the function)

Simplify: (attention: just perform the indicated operation...)

4) $\frac{3}{x^2-5x+6}+\frac{2}{x^2-4} = \frac{5x}{(x-3)(x^2-4)}$

5) $\frac{x^2}{x-1}\cdot\frac{x+1}{x+2}\div\frac{x}{(x-1)(x+2)} = \frac{x^{\cancel{2}}}{\cancel{x-1}}\cdot\frac{x+1}{\cancel{x+2}}\cdot\frac{\cancel{(x-1)}\cancel{(x+2)}}{\cancel{x}} = x(x+1)$

6) $\frac{\frac{x^2-y^2}{x+y}}{x^4-y^4} = \frac{\cancel{(x+y)}(x-y)}{\cancel{x+y}} \div (x^2-y^2)(x^2+y^2) = \cancel{x-y}\cdot\frac{1}{(x+y)\cancel{(x-y)}(x^2+y^2)}$

$= \frac{1}{(x+y)(x^2+y^2)}$

Find the constants A and B that make the equation true.

7) $\frac{2x-9}{x^2-x-6}=\frac{A}{x-3}+\frac{B}{x+2}$

$2x-9 = A(x+2) + B(x-3)$

$2x-9 = (A+B)x + 2A - 3B$

$A+B = 2$
$2A-3B = -9$

$\Longrightarrow$

$3A+3B = 6$
$2A-3B = -9$

$5A = -3$

$\boxed{A = -\frac{3}{5}}$ $\boxed{B = \frac{13}{5}}$

Solve the fractional equations. If it has no solution, say so...

8) $\frac{3}{x+1} - \frac{1}{x-2} = \frac{1}{x^2-x-2}$

$\frac{3}{x+1} - \frac{1}{x-2} = \frac{1}{(x-2)(x+1)}$

$3(x-2) - 1(x+1) = 1$

$2x - 7 = 1$

$2x = 8$ $\Rightarrow \boxed{x=4}$

9) $\frac{5}{a^2+a-6} = 2 - \frac{a-3}{a-2}$

$\frac{5}{(a+3)(a-2)} = \frac{2(a-2) - (a-3)}{(a-2)}$

$\frac{5}{(a+3)(a-2)} = \frac{a-1}{a-2}$

$5(a-2) = (a-1)(a+3)(a-2)$

$5a-10 = (a-1)(a^2+a-6)$

$5a-10 = a^3 + \cancel{a^2} - 6a - \cancel{a^2} - a + 6$

$5a-10 = a^3 - 7a + 6$

$a^3 - 12a + 16 = 0$

$(a-2)(a^2+2a-8) = 0$ $(a-2)(a+4)(a-2)$

$\underbrace{a=2}_{\text{not possible}}, \boxed{a=-4}$

EQUATIONS CONTAINING RADICALS

Example 1: Solve

$3x - 5\sqrt{x} = 2$

$3x - 2 = 5\sqrt{x}$ isolate the radical term

$9x^2 - 12x + 4 = 25x$ square both sides

$9x^2 - 37x + 4 = 0$ solve for x

$(x-4)(9x-1) = 0$

$x = 4$ or $x = \frac{1}{9}$ these are candidate solutions!

Now check each candidate above by plugging them into the original equation. The only possible solution is $x = 4$.

PRACTICE PROBLEMS: Solve. If an equation has no solution, say so. (hint: sometimes you may have to square it twice...)

1) $\sqrt{2x+5} - 1 = x$

$(\sqrt{2x+5})^2 = (x+1)^2$

$2x+5 = x^2+2x+1$

$x^2 = 4$

$x = \pm 2$

$\boxed{x=2}$ b/c $x=-2$ is not a solution!

2) $\sqrt{x-1} + \sqrt{x+4} = 5$

$(\sqrt{x-1})^2 = (5 - \sqrt{x+4})^2$ (squaring both sides)

$x-1 = 25 - 10\sqrt{x-4} + x + 4$

$-30 = -10\sqrt{x+4}$

$3 = \sqrt{x+4}$

$9 = x+4$ $\Rightarrow \boxed{x=5}$ ✓

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FUNCTION OPERATIONS AND COMPOSITION: INVERSE FUNCTIONS

What you need to know:

- The **composition** of a function g with a function f is defined as:
  $h(x) = g(f(x))$
  The domain of h is the set of all x-values such that x is in the domain of f and f(x) is in the domain of g.
- Functions f and g are inverses of each other provided:
  $f(g(x)) = x$ and $g(f(x)) = x$
  The function g is denoted as f^{-1}, read as "f inverse".
- When a function f has an inverse, for every point (a,b) on the graph of f, there is a point (b,a) on the graph of f^{-1}. This means that the graph of f^{-1} can be obtained from the graph of f by changing every point (x,y) on f to the point (y,x). This also means that the graphs of f and f^{-1} are reflections about the line y=x. Verify that the functions $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses!
- Horizontal Line Test: If any horizontal line which is drawn through the graph of a function f intersects the graph no more than once, then f is said to be a **one-to-one** function and has an inverse.

Let's try some problems!

Let f and g be functions whose values are given by the table below. Assume g is one-to-one with inverse g^{-1}.

X	f(x)	g(x)
1	6	2
2	9	3
3	10	4
4	-1	6

1) $f(g(3))$ -1

2) $g^{-1}(4) = 3$

3) $f(g^{-1}(6)) = -1$

4) $f^{-1}(f(g(2))) = 3$

5) $g(g^{-1}(2)) = 2$

* y-intercept = 23
means that when x = 0
y = 23. So...
$2(3(0)+a)^2+5 = 23$

6) If $f(x) = 2x^2 + 5$ and $g(x) = 3x + a$, find a so that the graph of $f \circ g$ crosses the y-axis at 23.

$f(g(x)) = f(3x+a) = 2(3x+a)^2 + 5$

$2a^2 + 15 = 23$

$2a^2 = 18$ $\boxed{a = \pm 3}$

TRIGONOMETRY

What you need to know:

- Trig values for selected angles:

Radians	Degrees	Sin x	Cos x	Tan x
0	0°	0	1	0
$\pi/6$	30°	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	90°	1	0	Und.

- Key Trig Identities:

(i) $\sin^2\theta + \cos^2\theta = 1$
(ii) $\tan^2\theta + 1 = \sec^2\theta$
(iv) $1 + \cot^2\theta = \csc^2\theta$
(v) $\sin(x+y) = \sin x\cos y + \cos x\sin y$
(vi) $\sin(x-y) = \sin x\cos y - \cos x\sin y$
(vii) $\cos(x+y) = \cos x\cos y - \sin x\sin y$
(viii) $\cos(x-y) = \cos x\cos y + \sin x\sin y$
(ix) $\sin 2\theta = 2\sin\theta\cos\theta$
(x) $\cos 2\theta = \cos^2\theta - \sin^2\theta$ or
$= 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$

Without using a calculator or table, find each value:

1. $\cos(-\pi)$ -1
2. $\sin\frac{3\pi}{2}$ -1
3. $\cos\frac{7\pi}{6}$ $-\frac{\sqrt{3}}{2}$
4. $\sin\frac{11\pi}{6}$ $-\frac{1}{2}$
5. $\cos\frac{5\pi}{4}$ $-\frac{\sqrt{2}}{2}$
6. $\sin\left(-\frac{2\pi}{3}\right)$ $-\frac{\sqrt{3}}{2}$

7. Find the slope and equation of the line with y-intercept = 4 and *inclination* 45°. (the *inclination* of a line is the angle α, where $0° \le \alpha \le 180°$, that is measured from the positive x-axis to the line. If a line has slope m, then $m = \tan\alpha$).

$m = 1$ through (0,4)

$\boxed{y = x + 4}$

8. Find the inclination of the line given the equation: 3x + 5y = 8 (you'll need a calculator here...)

$5y = -3x + 8$

$y = -\frac{3}{5}x + \frac{8}{5}$

$\tan\alpha = -\frac{3}{5} \Longrightarrow \alpha = \tan^{-1}\left(\frac{3}{5}\right)$

$\boxed{\alpha = 149°}$

Solve the trigonometric equations algebraically by using identities and without the use of a calculator. You must find all solutions in the interval $0 \le \theta \le 2\pi$:

9. $2\sin^2\theta - 1 = 0$

$\sin^2\theta = \frac{1}{2}$

$\sin\theta = \frac{\sqrt{2}}{2}$

$\theta = \pi/4,\ 3\pi/4$

10. $\cos x \tan x = \cos x$

$\cancel{\cos x}\,\frac{\sin x}{\cancel{\cos x}} = \cos x$

$\sin x = \cos x \Leftrightarrow x = \frac{\pi}{4},\ \frac{5\pi}{4}$

11. $\sin^2 x = \sin x$

$\sin^2 x - \sin x = 0$

$\sin x(\sin x - 1) = 0$

$\sin x = 0$ or $\sin x = 1$

$\boxed{x = 0, \frac{\pi}{2}, \pi, 2\pi}$

12. $2\cos^2 x = \cos x$

$\cos x(2\cos x - 1) = 0$

$\cos x = 0$ or $\cos x = \frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\Rightarrow$ $x = \frac{\pi}{3}, \frac{5\pi}{3}$

13. $2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$\cos x = \frac{1}{2}$ or $\cos x = -1$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ $\Rightarrow$ $\boxed{x = \pi}$

14. $\sin 2x = \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ $\Rightarrow$ $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\Rightarrow$ $x = \frac{\pi}{6}, \frac{5\pi}{6}$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

LOGARITHMS – What you need to know…

- Definition of Logarithm with base b: For any positive numbers b and y with $b \neq 1$, we define the logarithm of y with base b as follows:
  $$\log_b y = x \text{ if and only if } b^x = y$$
- LAWS OF LOGARITHMS (for $M, N, b > 0,\ b \neq 1$)
  - (i) $\log_b MN = \log_b M + \log_b N$
  - (ii) $\log_b \frac{M}{N} = \log_b M - \log_b N$
  - (iii) $\log_b M = \log_b N$ *if and only if* $M = N$
  - (iv) $\log_b M^k = k \cdot \log_b M$
  - (v) $b^{\log_b M} = M$
- The logarithmic and exponential functions are inverse functions. Example: Consider $f(x) = 2^x$ and $g(x) = \log_2 x$. Verify that (3,8) is on the graph of f and (8,3) on the graph of g. In addition,
  $$f(g(x)) = 2^{\log_2 x} = x \quad and \quad g(f(x)) = \log_2(2^x) = x$$ which verifies that f and g are indeed inverse functions.

Evaluate. The first is to be used as an example. (No calculators, please!)

1. $\log_4 64 = 3$ (because $4^3 = 64$)

2. $\log_5 \frac{1}{125} = -3$

3. $\log_4 \sqrt[4]{4} = \frac{1}{4}$

4. $\log_{\frac{1}{3}} 9 = -2$

5. $\log_{\sqrt{5}} 125 = 6$

6. $\log_{81} 243 = \frac{5}{4}$ (solve $81^x = 243 \Rightarrow 3^{4x} = 3^5 \Rightarrow 4x = 5 \Rightarrow \boxed{x = \frac{5}{4}}$

Solve for x.

7. $\log_x 64 = 3$
$x = 4$

8. $\log_3(x^2 - 7) = 2$
$\boxed{x = \pm 4}$

9. $\log_5(\log_3 x) = 0$
$x = 3$

10. $\log_6(\log_4(\log_2 x)) = 0$
$\log_4(\log_2 x) = 6^0 = 1 \Rightarrow \log_2 x = 4$
$\Rightarrow \boxed{x = 16}$

11. $\log_2(\log_x 64) = 1$
$\log_x 64 = 2 \Rightarrow \boxed{x = 8}$

12. Show that $\log_2 4 + \log_2 8 = \log_2 32$ by evaluating the 3 logarithms. Make a generalization based on your findings.
$\log_2 4 = 2$
$\log_2 8 = 3 \Rightarrow 2 + 3 = 5$
$\log_2 32 = 5$
Thus, The sum of 2 logs = log of a Product
or $\log_a A + \log_a B = \log_a A \cdot B$

Write the given expression as a rational number or as a single logarithm. (Use the Laws of Logarithms above!)

13. $\log 8 + \log 5 - \log 4$
$\log 10 = \textcircled{1}$

14. $\log_2 48 - \frac{1}{3}\log_2 27 = \log_2 16 = \textcircled{4}$

15. $\frac{1}{3}(2\log M - \log N - \log P)$
$\frac{1}{3}\left(\log \frac{M^2}{NP}\right)$

16. $\log_8 \sqrt{80} - \log_8 \sqrt{5} = \log_8 \sqrt{16}$
$= \boxed{\log_8 4}$

If $\log_8 3 = r$ *and* $\log_8 5 = s$, express the given logarithm in terms of r and s.

17. $\log_8 75 = r + 2s$
use $\log_8 75 = \log_8(3 \cdot 25)$

18. $\log_8 0.12 = r - 2s$
use: $\log_8 \frac{12}{100} = \log_8 \frac{3}{25} = \log_8 3 - \log_8 5^2$

19. Solve for x: $\log_2(x^2 + 8) = \log_2 x + \log_2 6$
$\log_2(x^2+8) = \log_2 6x \Rightarrow x^2 + 8 = 6x \Rightarrow x^2 - 6x + 8 = 0 \Rightarrow$
$\Rightarrow (x-4)(x-2) = 0 \Rightarrow \boxed{x = 4}$ or $\boxed{x = 2}$

Solve (no calculator please!)

20. $3^x = 81$
$x = 4$
Use the fact $3^x = 3^4$

21. $4^x = 8$
$\boxed{x = \frac{3}{2}}$
change into: $2^{2x} = 2^3$

22. $4^x = 16\sqrt{2}$
$\boxed{x = \frac{9}{4}}$
change into $2^{2x} = 2^{\frac{9}{2}}$

23. $2^{2x}-2^{x}-6=0$ (hint: $2^{2x}-2^{x}-6=(2^{x})^{2}-(2^{x})-6$) → Treat this as a quadratic equation: use the substitution $2^x = u$)

$\boxed{x = \frac{\log 3}{\log 2}}$

change into: $u^2 - u - 6 = 0$

$(u-3)(u+2) = 0 \Rightarrow$

$u = 3$ or $\boxed{u = -2}$ not possible

If $u = 3 \Rightarrow 2^x = 3 \Rightarrow \log 2^x = \log$

$\Rightarrow x = \frac{\log 3}{\log 2}$

Simplify (recall: the notation ln stands for the natural logarithm base e)

24. a) $\ln e^{x} = x$ b) $e^{\ln x} = x$ c) $e^{2\ln x} = x^2$ d) $e^{-\ln x} = x^{-1}$ or $\frac{1}{x}$

Solve for x (answers can be left in terms of e).

25. $\ln x-\ln(x-1)=\ln 2$

$\boxed{x = 2}$

Solve: $\ln \frac{x}{x-1} = \ln 2$

26. $(\ln x)^{2}-6(\ln x)+9=0$

Let $u = \ln x$

$u^2 - 6u + 9 = 0 \Rightarrow (u-3)^2 = 0$

$u = 3 \Rightarrow \ln x = 3$ $\boxed{x = e^3}$

We conclude with the following definition:

EXPONENTIAL GROWTH MODEL:

Consider an initial quantity Y_0 that changes exponentially with time t. At any time t the amount Y after t units of time may be given by:

$$Y = Y_0 e^{kt}$$

Where k>0 represents the **growth rate** and k<0 represents the **decay rate**.

27. **Growth of Cholera**

Suppose that the cholera bacteria in a colony grows according to the exponential model $P = P_0 e^{kt}$). The colony starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 hours?

Step 1: Find the growth rate k

$2 = 1 \cdot e^{.5k}$

$\ln 2 = .5k \Rightarrow \boxed{k = \frac{\ln 2}{.5}}$

Step 2:

$P = e^{\frac{\ln 2}{.5}(24)}$

$\boxed{P = 2.815 \times 10^{14}}$ ← number of bacteria

28. **Bacteria Growth**

A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours

there are 10,000 bacteria. At the end of 5 h there are 40,000 bacteria. How many bacteria were present initially?

From condition 1: $10{,}000 = P_0 e^{3k} \Rightarrow P_0 = \frac{10000}{e^{3k}}$ ⓐ

From condition 2: $40{,}000 = P_0 e^{5k} \Rightarrow P_0 = \frac{40000}{e^{5k}}$ ⓑ

Thus: $\frac{10000}{e^{3k}} = \frac{40000}{e^{5k}} \Rightarrow \frac{e^{5k}}{e^{3k}} = \frac{40000}{10000} \Rightarrow e^{2k} = 4 \Rightarrow 2k = \ln 4 \Rightarrow \boxed{k = \frac{\ln 4}{2}}$

* Substituting in either ⓐ

$P_0 = \frac{10000}{e^{\frac{3}{2}\ln 4}} = 1250$ bacte

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. The next problem shows that the half-life is a **constant** that depends only on the radioactive substance and not on the number of radioactive nuclei present in the sample.

29) Find the half-life of a radioactive substance with decay equation $y = y_0 e^{-kt}$ and show that the half-life depends only on k. (hint: set up the equation $y_0 e^{-kt} = \frac{1}{2} y_0$. Solve it algebraically for t!)

Solve $\frac{1}{2} = e^{-k\tau} \Rightarrow \ln \frac{1}{2} = -k\tau \Rightarrow \tau = \left[\ln \frac{1}{2}\right]\left(-\frac{1}{k}\right) \Rightarrow$

$\boxed{\tau = \frac{\ln 2}{k}}$

Conclusion: The half-life of a radioactive substance with rate constant k (k>0) is

Half-life = $\frac{\ln 2}{k}$

30) **Using Carbon-14 Dating**

Scientists who do carbon-14 dating use 5700 years for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Since $5700 =$ half-life $\Rightarrow 5700 = \frac{\ln 2}{k} \Rightarrow k = \frac{\ln 2}{5700}$

10% decayed ⇒ 90% of radiactive nuclei still present

Thus, solve $.9 = e^{\frac{\ln 2}{5700}\tau}$

$\ln .9 = \frac{\ln 2}{5700}\tau$

$\tau = \frac{5700}{\ln 2} \cdot \ln(0.9)$

$\tau = -866.4$ years

age is 866.4 years

8. If $\lim_{x\to c} f(x) = -2$ and $\lim_{x\to c} g(x) = 5$, determine the following:

a. $\lim_{x\to c} 5f(x)$

$5(-2) = -10$

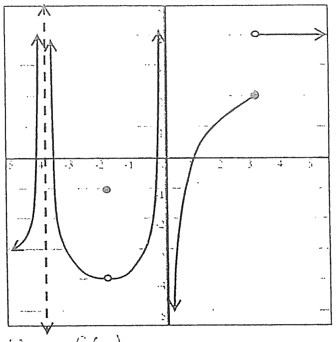
b. $\lim_{x\to c} f(x) - 3g(x)$

$(-2) - 3(5) = -17$

c. $\lim_{x\to c} \frac{f(x)^3}{\sqrt[3]{g(x)}}$

$\frac{(-2)^3}{\sqrt[3]{5}} = -\frac{8}{\sqrt[3]{5}}$

9. Let f(x) be the graph below. Determine the following:



a. f(-2) = -1

b. $\lim_{x\to -2^{+}} f(x)$ = -4

c. $\lim_{x\to -2^{+}} f(x)$ = -4

d. $\lim_{x\to -2} f(x)$ = -4

e. $\lim_{x\to 3^{-}} f(x)$ = 2

f. $\lim_{x\to 3^{+}} f(x)$ = 4

g. $\lim_{x\to 3} f(x)$ = DNE

h. f(3) = 2

i. Is f continuous 3? No b/c $\lim_{x\to 3^{+}} f(x) \neq \lim_{x\to 3^{-}} f(x)$

j. $\lim_{x\to -4^{-}} f(x)$ ∞

k. $\lim_{x\to -4^{+}} f(x) = \infty$

l. $\lim_{x\to -4} f(x) = \infty$

m. f(-4) = DNE

n. Is f continuous at -4? No b/c f(-4) DNE

o. $\lim_{x\to 0^{-}} f(x)$

∞

p. $\lim_{x\to 0^{+}} f(x)$

$-\infty$

q. $\lim_{x\to 0} f(x)$

DNE b/c $\lim_{x\to 0^{+}} f(x) \neq \lim_{x\to 0^{-}} f(x)$

r. f(0) DNE

s. $\lim_{x\to -\infty} f(x)$

$-\infty$

t. $\lim_{x\to \infty} f(x)$

4

One-Sided limit: exists if graph approaches a value from one side: + (right) or – (left)

Evaluating Limits:

1. Tables: either from table or by graphing and viewing table looking for values of y approaching the same number. Check one sided limits first and if they are equal the overall limit exists.
2. Graphically: Check one sided limits (again – value of limit is the y-value or the height of the graph) and if they are equal overall limit exists.
3. Algebraically: plug in c. If you get a constant then limit exists. Rational functions may need to be simplified first!

Properties:

1. $\lim_{x\to c} A = A$: the limit of a constant A is the constant A- think of it as the graph of a horizontal line. Y = A independent of what the x value is.
2. $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$: you can separate sum and difference of terms.
3. $\lim_{x\to c} A(f(x)) = A \lim_{x\to c} f(x)$: you can pull out numeric factors.
4. $\lim_{x\to c}(f(x))^n = [\lim_{x\to c} f(x)]^n$ you can evaluate limit first, then raise your solution to the power.
5. $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ you can evaluate the limit of the numerator and denominator separately then evaluate their quotient.

Examples: Evaluate the limits:

1. $\lim_{x\to 3} 3x^2 + 3$

$= 3(3)^2 + 3 = 30$

2. $\lim_{x\to -4^{+}} \frac{3}{x^2-16} = \lim_{x\to -4^{+}} \frac{3}{(x+4)(x-4)}$

Check graph.

$\lim_{x\to -4^{+}} \frac{3}{x^2-16} = -\infty$

3. $\lim_{x\to 2} \sqrt[3]{x^3 - 1}$

$\sqrt[3]{(2)^3 - 1} = \sqrt[3]{7}$

4. $\lim_{x\to \pi} x\cos x$

$= \pi \cdot \cos\pi = -\pi$

5. $\lim_{x\to -\infty} \sin x$

check graph:

$\lim_{x\to -\infty} \sin x$ = DNE

6. $\lim_{x\to 2^{-}} 5 = 5$

7. If $f(x) = \begin{cases} 2x - 3, & if\ x < -1 \\ 2, & if\ x = -1 \\ 5x, & if\ x > -1 \end{cases}$, determine the following:

a. $\lim_{x\to -1^{-}} f(x)$

$= 2(-1) - 3 = -5$

b. $\lim_{x\to -1^{+}} f(x)$

$5(-1) = -5$

c. $\lim_{x\to -1} f(x)$

$\lim_{x\to -1^{+}} = \lim_{x\to -1^{-}} = -5$

$\therefore \lim_{x\to -1} f(x) = -5$

d. Is f continuous?

No b/c

$f(-1) = 2 \neq -5$