SUMMER PREP

FOR

AP CALCULUS AB/BC

Topics from Algebra and Pre-Calculus

(Key contains solved problems)

Note: The purpose of this packet is to give you a review of basic skills. You are asked **not to use the calculator**, except on p.13 (8) and p.16-17(27-30all).

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ALGEBRA TOPICS

LAWS OF EXPONENTS

Assume **a** and **b** are real numbers and **m** and **n** positive integers.

(i) $a^m \cdot a^n = a^{m+n}$ (ii) $(ab)^m = a^m \cdot b^m$ (iii) $(a^m)^n = a^{mn}$

(iv) $a^{-m} = \dfrac{1}{a^m}$ (v) $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

PRACTICE PROBLEMS (NO , please!)

Simplify:

1) $\left(2c^2d^3\right)^3$

2) $(a^2b)(-3ab^3)(-2ab)^{-2}$

3) $r^{h-2}(r^{h+1})^2$

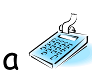
4) Solve for n: $3^{5n} = 3^5(3^{2n})^2$

5) $4^{n+3} \cdot 16^n = 8^{3n}$

6) Write in exponential form (no negative exponents): $\sqrt[3]{8x^6y^{-4}}$

7) Write in simplest radical form: $\sqrt[4]{27} \cdot \sqrt[8]{9}$

8) Simplify. Give your answer in exponential form: $a^{\frac{1}{2}}\left(a^{\frac{3}{2}} - 2a^{\frac{1}{2}}\right)$

9) Using the property of exponents, show that $27^{\frac{4}{3}} - 9^{\frac{3}{2}} = 2 \cdot 3^3$ (without the help of a , of course!)

Solve (algebraic solution!):

10) $(3x+1)^{\frac{3}{4}} = 8$

11) $9x^{-\frac{2}{3}} = 4$

FACTORING POLYNOMIALS

- **Perfect Square Trinomials**: $a^2 + 2ab + b^2 = (a+b)^2$
  $a^2 - 2ab + b^2 = (a-b)^2$
- **Difference of Squares**: $a^2 - b^2 = (a+b)(a-b)$
- **Sum of Cubes**: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- **Difference of Cubes**: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

PRACTICE PROBLEMS – Factor completely:

1) $3x^2 - x - 10$

2) $4x^2 + 12x - 7$

3) $a^2 - 10a + 25$

4) $4a^2 - 4ab + b^2$

5) $25x^2 - 16y^2$

6) $3x^5 - 48x$

7) $x^6 - y^6$

8) $64 - z^6$

9) $8p^3 + 1$

10) $x(y-2) + 3(2-y)$

11) $x^2 - 6x + 9 - 4y^2$

12) $(x+y)^3 + (x-y)^3$

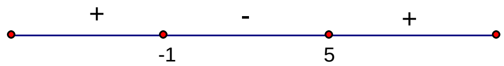
13) $6x^2 - 7xy - 3y^2$

14) $4x^3 + 8x^2y - 5xy^2$

SOLVING POLYNOMIAL INEQUALITIES

Example 1: $x^2 - 4x - 5 < 0$

Solution: $(x+1)(x-5) < 0$ *factor*



mark the zeros; pick a test point to determine the sign of the polynomial in each interval- this is called a ***SIGN CHART!***

(-1, 5)

PRACTICE PROBLEMS

1) $x^3 + 7x^2 + 10x > 0$

2) $x^3 - 16x > 0$

3) $(x^2 - 4x)(x^2 - 4) \leq 0$

4) $x^2 - x - 12 < 0$

PRE-CALCULUS TOPICS

FUNCTIONS

You are given a polynomial equation and one or more of its roots. Find the remaining roots.
What you need to know:

(i) **The Remainder Theorem**: When a polynomial P(x) is divided by (x-a), the remainder is P(a);

(ii) **Factor Theorem**: For a polynomial P(x), (x-a) is a factor if and only if P(a)=0;

(iii) Use **synthetic** division when dividing a polynomial by a linear factor; long division will always work;

(iv) **Odd Function**: A function f(x) is odd $\Leftrightarrow$ f(-x) = -f(x);

(v) **Even Function**: A function f(x) is even $\Leftrightarrow$ f(-x) = f(x);
Remarks: item (iv) means the graph of the function has **rotational symmetry** about the origin, and (v) means the graph has **reflectional symmetry** about the y-axis;

Find the quotient and the remainder when the first polynomial is divided by the second. No calculators allowed!

1) $x^3 - 2x^2 + 5x + 1;\ x - 1$

2) $3x^4 - 2x^3 + 5x^2 + x + 1;\ x^2 + 2x$

3) Determine whether x-1 or x+1 are factors of $x^{100} - 4x^{99} + 3$.

4) When a polynomial P(x) is divided by $3x - 4$, the quotient is $x^3 + 2x + 2$ and the remainder is -1. Find P(x). (*recall: $P(x) = (\text{divisor}) \times (\text{quotient}) + \text{remainder}!$*)

5) You are given a polynomial equation and one or more of its roots. Find the remaining roots.
$2x^4 - 9x^3 + 2x^2 + 9x - 4 = 0$; roots: x = -1, x = 1.

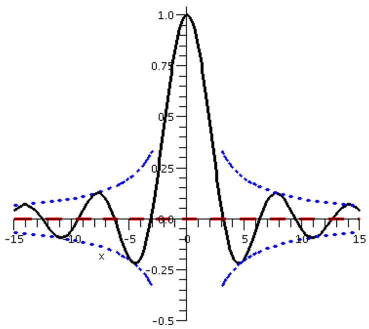
Verify algebraically whether the functions are odd or even.

6) $f(x) = 5x^3 - x$ 7) $h(x) = \dfrac{5}{x^2 + 1}$

FINDING THE ASYMPTOTES OF A RATIONAL FUNCTION

What you need to know:

- **Vertical Asymptote**: Set denominator = 0 and solve it. For values of x near the asymptote, the y-values of the function "approach" infinity (i.e., the y-values get increasingly large…)
- **Horizontal** or **Oblique Asymptote**: when the values of x get very large (approach infinity), the y-values tend to the asymptote. Below is an example of a curve displaying asymptotic behavior:



A curve can intersect its asymptote, even infinitely many times. This **does not apply** to vertical asymptotes of functions.

- To find the horizontal or oblique asymptote, divide the numerator by the denominator. The asymptote is given by the quotient function.

PRACTICE PROBLEMS (No , please!)

For the functions below determine: a) the domain; b) the x-and y-intercepts; c) the vertical, horizontal or oblique asymptotes; and d) whether R(x) is even or odd (show algebraic analysis).

1) $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

2) $r(x) = \dfrac{3x^2 + 5x - 2}{x^2 - 4}$. *(explain what happens at x=-2…)*

3) Consider $f(x) = \frac{8}{2+x^2}$. a) is f even or odd? b) explain why there are no vertical asymptotes; c) What is the domain of f? Range? d) Given the equation of its horizontal asymptote.

4) Find a quadratic function f(x) with y-intercept -2 and x-intercepts (-2,0) and $\left(\frac{1}{3}, 0\right)$.

RATIONAL ALGEBRAIC EXPRESSIONS AND FUNCTIONS

What you need to know:

(i) A rational function is a function of the form $\frac{p(x)}{q(x)}$, where **p** and **q** are polynomial functions and $q(x) \neq 0$ for every x.

(ii) To **simplify** a rational expression, factor both numerator and denominator. Cancel out common factors.

(iii) To find the **domain** of a rational function: solve q(x) = 0. The zeroes of the polynomial q(x) must be excluded from the domain.

(iv) To find the **zeroes** of a rational function: set p(x) = 0 and solve it. The zeros of p(x) are the zeros of the polynomial function (provided they are not also zeros of the denominator q(x)!)

In #1-3: Find (a) the domain of each function; and (b) the zeros, if any. No calculators, please!

1) $g(x) = \frac{2x^2 + 3x - 9}{x^3 - 4x}$

2) $g(x) = \dfrac{(x+3)(x-2)(x-1)}{x^2 - 6x + 8}$

3) $h(x) = \dfrac{x^3 + x^2 - x - 1}{x^3 - x^2 - x + 1}$

Simplify: (*attention: just perform the indicated operation…*)

4) $\dfrac{3}{x^2 - 5x + 6} + \dfrac{2}{x^2 - 4}$

5) $\dfrac{x^2}{x-1} \cdot \dfrac{x+1}{x+2} \div \dfrac{x}{(x-1)(x+2)}$

6) $\dfrac{\dfrac{x^2 - y^2}{x + y}}{x^4 - y^4} =$

Find the constants A and B that make the equation true. FRACTION DECOMPOSITION!

7) $\dfrac{2x - 9}{x^2 - x - 6} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$

Solve the fractional equations. If it has no solution, say so…

8) $\frac{3}{x+1}-\frac{1}{x-2}=\frac{1}{x^2-x-2}$

9) $\frac{5}{a^2+a-6}=2-\frac{a-3}{a-2}$

EQUATIONS CONTAINING RADICALS

Example 1: Solve

$3x-5\sqrt{x}=2$

$3x-2=5\sqrt{x}$ *isolate the radical term*

$9x^2-12x+4=25x$ *square both sides*

$9x^2-37x+4=0$ *Solve for x*

$(x-4)(9x-1)=0$

$x=4 \text{ or } x=\frac{1}{9}$ *these are* ***candidate*** *solutions!*

Now check each candidate above by plugging them into the original equation. The only possible solution is $x=4$.

PRACTICE PROBLEMS: Solve. If an equation has no solution, say so. *(hint: sometimes you may have to square it twice…)*

1) $\sqrt{2x+5}-1=x$

2) $\sqrt{x-1}+\sqrt{x+4}=5$

FUNCTION OPERATIONS AND COMPOSITION; INVERSE FUNCTIONS

What you need to know:

- The **composition** of a function ***g*** with a function ***f*** is defined as:

  $$h(x) = g(f(x))$$

  The domain of h is the set of all x-values such that x is in the domain of f and f(x) is in the domain of g.
- Functions f and g are inverses of each other provided:

  $$f(g(x)) = x \quad and \quad g(f(x)) = x$$

  The function g is denoted as f^{-1}, read as "f inverse".
- When a function f has an inverse, for every point (a,b) on the graph of f, there is a point (b,a) on the graph of f^{-1}. This means that the graph of f^{-1} can be obtained from the graph of f by changing every point (x,y) on f to the point (y,x). This also means that the graphs of f and f^{-1} are reflections about the line y=x. Verify that the functions $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses!
- Horizontal Line Test: If any horizontal line which is drawn through the graph of a function f intersects the graph no more than once, then f is said to be a **one-to-one** function and has an inverse.

Let f and g be functions whose values are given by the table below. Assume g is one-to-one with inverse g^{-1}.

X	f(x)	g(x)
1	6	2
2	9	3
3	10	4
4	-1	6

1) $f(g(3))$ 2) $g^{-1}(4)$ 3) $f(g^{-1}(6))$ 4) $f^{-1}(f(g(2)))$ 5) $g(g^{-1}(2))$

6) If $f(x) = 2x^2 + 5$ and $g(x) = 3x + a$, find a so that the graph of $f \circ g$ crosses the y-axis at 23.

TRIGONOMETRY

What you need to know:

- Trig values for selected angles:

Radians	Degrees	Sin x	Cos x	Tan x
0	0°	0	1	0
$\pi/6$	30°	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	90°	1	0	Und.

- Key Trig Identities – formulas you should know:

(*i*) $\sin^2\theta + \cos^2\theta = 1$

(*ii*) $\tan^2\theta + 1 = \sec^2\theta$

(*iv*) $1 + \cot^2\theta = \csc^2\theta$

(*v*) $\sin(x + y) = \sin x \cos y + \cos x \sin y$

(*vi*) $\sin(x - y) = \sin x \cos y - \cos x \sin y$

(*vii*) $\cos(x + y) = \cos x \cos y - \sin x \sin y$

(*viii*) $\cos(x - y) = \cos x \cos y + \sin x \sin y$

(*ix*) $\sin 2\theta = 2\sin\theta\cos\theta$

(*x*) $\cos 2\theta = \cos^2\theta - \sin^2\theta$ *or*

$= 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

Without using a calculator or table, find each value:

1. $\cos(-\pi)$

2. $\sin\frac{3\pi}{2}$

3. $\cos\frac{7\pi}{6}$

4. $\sin\frac{11\pi}{6}$

5. $\cos\frac{5\pi}{4}$

6. $\sin\left(-\frac{2\pi}{3}\right)$

7. Find the slope and equation of the line with y-intercept = 4 and ***inclination*** 45°. (the ***inclination*** of a line is the angle α, where $0^\circ \le \alpha \le 180^\circ$, that is measured from the positive x-axis to the line. If a line has slope m, then $m = \tan\alpha$).

8. Find the inclination of the line given the equation: 3x + 5y = 8 *(you'll need a calculator here…)*

Solve the trigonometric equations algebraically by using identities and **without** the use of a calculator. You must find all solutions in the interval $0 \le \theta \le 2\pi$:

9. $2\sin^2\theta - 1 = 0$

10. $\cos x \tan x = \cos x$

11. $\sin^2 x = \sin x$

12. $2\cos^2 x = \cos x$

13. $2\cos^2 x + \cos x - 1 = 0$

14. $\sin 2x = \cos x$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

LOGARITHMS – What you need to know…

- **Definition of Logarithm with base b**: For any positive numbers b and y with $b \neq 1$, we define the **logarithm of y with base b** as follows:

  $\log_b y = x$ if and only if $b^x = y$

- **LAWS OF LOGARITHMS** (for M, N, b > 0, $b \neq 1$)
  $(i)\ \log_b MN = \log_b M + \log_b N$
  $(ii)\ \log_b \frac{M}{N} = \log_b M - \log_b N$
  $(iii)\ \log_b M = \log_b N \text{ if and only if } M = N$
  $(iv)\ \log_b M^k = k \cdot \log_b M$
  $(v)\ b^{\log_b M} = M$

- The logarithmic and exponential functions are **inverse** functions.
  Example: Consider $f(x) = 2^x$ and $g(x) = \log_2 x$. Verify that (3,8) is on the graph of f and (8,3) on the graph of g. In addition,
  $f(g(x)) = 2^{\log_2 x} = x$ *and* $g(f(x)) = \log_2(2^x) = x$ which verifies that f and g are indeed inverse functions.

Evaluate. (No Calculators, please!)

1. $\log_5 \frac{1}{125} =$

2. $\log_2(\log_x 64) = 1$

3. $\log_4 \sqrt[4]{4} =$

4. $\log_{\frac{1}{3}} 9 =$

5. $\log_{\sqrt{5}} 125 =$

6. $\log_{81} 243 =$

Solve for x.

7. $\log_x 64 = 3$

8. $\log_3(x^2 - 7) = 2$

9. $\log_5(\log_3 x) = 0$

10. $\log_6(\log_4(\log_2 x)) = 0$

12. Show that $\log_2 4 + \log_2 8 = \log_2 32$ by evaluating the 3 logarithms. Make a generalization based on your findings.

Write the given expression as a rational number or as a single logarithm. (Use the laws of logarithms above!)

13. $\log 8 + \log 5 - \log 4$

14. $\log_2 48 - \frac{1}{3}\log_2 27$

15. $\frac{1}{3}(2\log M - \log N - \log P)$

16. $\log_8 \sqrt{80} - \log_8 \sqrt{5}$

If $\log_8 3 = r$ *and* $\log_8 5 = s$, express the given logarithm in terms of r and s.

17. $\log_8 75$

18. $\log_8 0.12$

19. Solve for x: $\log_2(x^2 + 8) = \log_2 x + \log_2 6$

Solve (no calculator please!)

20. $3^x = 81$

21. $4^x = 8$

22. $4^x = 16\sqrt{2}$

23. $2^{2x} - 2^x - 6 = 0$ (hint: $2^{2x} - 2^x - 6 = (2^x)^2 - (2^x) - 6$)

Simplify (recall: the notation $\ln$ stands for the natural logarithm base e)

24. $a)\ \ln e^x$ $\quad b)\ e^{\ln x}$ $\quad c)\ e^{2\ln x}$ $\quad d)\ e^{-\ln x}$

Solve for x (answers can be left in terms of e).

25. $\ln x - \ln(x-1) = \ln 2$

26. $(\ln x)^2 - 6(\ln x) + 9 = 0$

We conclude with the following definition:

EXPONENTIAL GROWTH MODEL:

Consider an initial quantity Y_o that changes exponentially with time t. At any time t the amount Y after t units of time may be given by:

$$Y = Y_o e^{kt}$$

Where k>0 represents the **growth rate** and k<0 represents the **decay rate**.

27. **Growth of Cholera:** Suppose that the cholera bacteria in a colony grows according to the exponential model $P = P_o e^{kt}$). The colony starts with 1 bacterium and doubles in number every half hour. How many bacteria **will** the colony contain at the end of 24 hours?

28. **Bacteria Growth:** A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 h there are 40,000 bacteria. How many bacteria were present initially?

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. The next problem shows that the half-life is **a constan**t that depends only on the radioactive substance and not on the number of radioactive nuclei present in the sample.

29) Find the half-life of a radioactive substance with decay equation $y = y_o e^{-kt}$ and show that the half-life depends only on k. (hint: set up the equation $y_o e^{-kt} = \frac{1}{2} y_0$. Solve it algebraically for t!)

30) **Using Carbon-14 Dating**

Scientists who do carbon-14 dating use 5700 years for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Limits: $\lim_{x\to c} f(x) = N$ "Limit as x approaches c of f(x) equals N"
LIMIT ≠ CONTINUITY
Understanding the difference between limits and continuity: Limit is the value (y) that the function APPROACHES as you get close to c (either from one side or from both sides). Continuity implies that the limit exists (from both sides) and that the limit = the value at c!

Limit Exists: $\mathbf{\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)}$
Continuous: $\mathbf{\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)\ AND = f(c) = A\ (where\ A\ is\ a\ constant\ not\ \pm\infty)}$

One-Sided limit: exists if graph approaches a value from one side: + (right) or – (left)
Evaluating Limits:

1. Tables: either from table or by graphing and viewing table looking for values of y approaching the same number. Check one sided limits first and if they are equal the overall limit exists.
2. Graphically: Check one sided limits (again – value of limit is the y-value or the height of the graph) and if they are equal overall limit exists.
3. Algebraically: plug in c. If you get a constant then limit exists. Rational functions may need to be simplified first!

Properties:

1. $\lim_{x\to c} A = A$: the limit of a constant A is the constant A- think of it as the graph of a horizontal line. Y = A independent of what the x value is.
2. $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$: you can separate sum and difference of terms.
3. $\lim_{x\to c} A\big(f(x)\big) = A \lim_{x\to c} f(x)$: you can pull out numeric factors.
4. $\lim_{x\to c}\big(f(x)\big)^n = [\lim_{x\to c} f(x)]^n$ you can evaluate limit first, then raise your solution to the power.
5. $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ you can evaluate the limit of the numerator and denominator separately then evaluate their quotient.

Examples: Evaluate the limits:

1. $\lim_{x\to 3} 3x^2 + 3$

2. $\lim_{x\to -4^+} \frac{3}{x^2-16}$

3. $\lim_{x\to 2} \sqrt[3]{x^3 - 1}$

4. $\lim_{x\to \pi} x\cos x$

5. $\lim_{x\to -\infty} \sin x$

6. $\lim_{x\to 2^-} 5$

7. If $f(x) = \begin{cases} 2x - 3, & if\ x < -1 \\ 2, & if\ x = -1 \\ 5x, & if\ x > -1 \end{cases}$, determine the following:

a. $\lim_{x\to -1^-} f(x)$

b. $\lim_{x\to -1^+} f(x)$

c. $\lim_{x\to -1} f(x)$

d. Is f continuous?

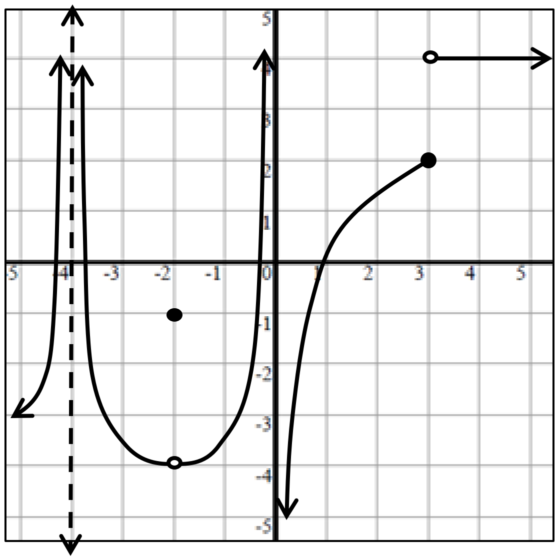
8. If $\lim_{x\to c} f(x) = -2$ and $\lim_{x\to c} g(x) = 5$, determine the following:

a. $\lim_{x\to c} 5f(x)$

b. $\lim_{x\to c}[f(x) - 3g(x)]$

c. $\lim_{x\to c} \frac{f(x)^3}{\sqrt[3]{g(x)}}$

9. Let f(x) be the graph below. Determine the following:



a. f(-2)

b. $\lim_{x\to -2^+} f(x)$

c. $\lim_{x\to -2^-} f(x)$

d. $\lim_{x\to -2} f(x)$

e. $\lim_{x\to 3^-} f(x)$

f. $\lim_{x\to 3^+} f(x)$

g. $\lim_{x\to 3} f(x)$

h. f(3)

i. Is f continuous 3?

j. $\lim_{x\to -4^-} f(x)$

k. $\lim_{x\to -4^+} f(x)$

l. $\lim_{x\to -4} f(x)$

m. f(-4)

n. Is f continuous at -4?

o. $\lim_{x\to 0^-} f(x)$

p. $\lim_{x\to 0^+} f(x)$

q. $\lim_{x\to 0} f(x)$

r. f(0)

s. $\lim_{x\to -\infty} f(x)$

t. $\lim_{x\to \infty} f(x)$