AP Calculus Summer Prep

Topics from Algebra and Pre-Calculus

(Solutions are on the Answer Key on the Last 2 Pages)

The purpose of this packet is to give you a review of basic skills. You are asked to have this packet completed for the first day of school and **to not use as graphing calculator** unless noted. Questions will be answered on the first day of class. Please note those problems that were challenging in the space below.

NOTES:

ALGEBRA TOPICS

FACTORING POLYNOMIALS

- **Perfect Square Trinomials:** $a^2 + 2ab + b^2 = (a+b)^2$
  $a^2 - 2ab + b^2 = (a-b)^2$
- **Difference of Squares:** $a^2 - b^2 = (a+b)(a-b)$
- **Sum of Cubes:** $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- **Difference of Cubes:** $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

PRACTICE PROBLEMS – Factor completely:

1) $3x^2 - x - 10$

2) $25x^2 - 16y^2$

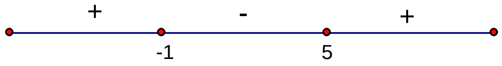
3) $x(y-2) + 3(2-y)$

4) $2x^{\left(\frac{1}{2}\right)}(x+2) + 2x^{\left(\frac{3}{2}\right)}(x+2)^{-1}$

SOLVING POLYNOMIAL INEQUALITIES

Example 1: $x^2 - 4x - 5 < 0$

Solution: $(x+1)(x-5) < 0$ *factor and determine critical values: x = -1, 5*



Answer: (-1, 5) *mark the zeros; pick a test point to determine the sign of the polynomial in each interval- this is called a* ***SIGN CHART!***

PRACTICE PROBLEMS – Include a Sign Chart

5) $x^3 + 7x^2 + 10x > 0$

6) $x^3 \leq 16x$

PRE-CALCULUS TOPICS

Analyzing Graphs:
Domain and Range, x-intercepts (zeros) and y-intercepts, extrema (local and absolute)
End Behavior: $\lim_{x\to\pm\infty} f(x)$
Continuity: removable, non-removable, All Polynomials are continuous for all x.
Even/Odd Functions:
Even: Symmetric to the y-axis. Algebraically: f(-x) = f(x)
Odd: Symmetric to the origin. Algebraically: f(-x) = f(x)

Analyze the following functions without graphing them on a graphing calculator (except where noted).

7) $f(x) = 2x^2 - 20x + 57$

Domain:

Range:

x-intercept(s):

y-intercept(s):

Extrema (by hand):

End Behavior:

$\lim_{x\to-\infty} f(x) =$

$\lim_{x\to\infty} f(x) =$

Interval of Continuity:

Tests for Symmetry:

8) $g(x) = 3x^3 - 6x$

Domain:

Range:

x-intercept(s):

y-intercept(s):

Extrema (on calc):

End Behavior:

$\lim_{x\to-\infty} g(x) =$

$\lim_{x\to\infty} g(x) =$

Interval of Continuity:

Tests for Symmetry:

Rational Expressions and Functions:
Domain and Range
End Behavior: $\lim_{x\to\pm\infty} f(x)$
Asymptotes: vertical and horizontal asymptotes
Simplify

9) Consider $f(x)=\dfrac{8}{2+x^2}$.

Domain:

Range:

x-intercept(s):

y-intercept(s):

End Behavior:

$\lim_{x\to-\infty} f(x) =$

$\lim_{x\to\infty} f(x) =$

Interval of Continuity:

Tests for Symmetry:

10 – 13 Simplify.

10) $h(x)=\dfrac{x^3+x^2-x-1}{x^3-x^2-x+1}$

11) $\dfrac{3}{x^2-5x+6}+\dfrac{2}{x^2-4}$

12) $\dfrac{\dfrac{x^2-y^2}{x+y}}{x^4-y^4}=$

13) $\dfrac{5x(x^2+1)^3(x+2)^{-1}+10(x+2)(x^2+1)^2}{20(x+2)(x^2+1)^3}$

FUNCTION OPERATIONS AND COMPOSITION; INVERSE FUNCTIONS

Composition of a function *g* with a function *f* is defined as:

$$h(x) = g(f(x))$$

The domain of h is the set of all x-values such that x is in the domain of f and f(x) is in the domain of g.

Inverses: Functions f and g are inverses of each other provided:

$$f(g(x)) = x \quad and \quad g(f(x)) = x$$

The function g is denoted as f^{-1}, read as "f inverse".

Horizontal Line Test: If any horizontal line which is drawn through the graph of a function f intersects the graph no more than once, then f is said to be a **one-to-one** function and has an inverse.

Let f and g be functions whose values are given by the table below. Assume g is one-to-one with inverse g^{-1}.

X	f(x)	g(x)
1	6	2
2	9	3
3	10	4
4	-1	6

14) $f(g(3))$

15) $g^{-1}(4)$

16) $f(g^{-1}(6))$

17) $f^{-1}(f(g(2)))$

18) $g(g^{-1}(2))$

19) If $f(x) = x^2 - 2x + 3$ and g(x) is the inverse of f(x). Determine g(2).

TRIGONOMETRY

Trig values for selected angles:

Radians	Degrees	Sin x	Cos x	Tan x
0	0°	0	1	0
$\pi/6$	30°	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	90°	1	0	Und.

Key Trig Identities – formulas you should know:

$$\sin^2 x + \cos^2 x = 1$$
$$\sin(2x) = 2\sin x \cos x$$
$$\cos(2x) = 1 - 2\sin^2 x$$
$$1 + \tan^2 x = \sec^2 x$$
$$\sin(-x) = -\sin x \quad ODD$$
$$\cos(-x) = \cos x \quad EVEN$$

Without using a calculator or table, find each value:

20) $\cos\left(-\frac{\pi}{3}\right)$

21) $\sin\left(\frac{11\pi}{6}\right)$

22) $\cos\left(\frac{5\pi}{4}\right)$

Solve the trigonometric equations algebraically by using identities and **without** the use of a calculator. You must find all solutions in the interval $0 \le \theta \le 2\pi$:

23) $2\sin^2\theta - 1 = 0$

24) $\cos x \tan x = \cos x$

25) $2\cos^2 x + \cos x - 1 = 0$

26) $\sin 2x = \cos x$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Logarithm: For any positive numbers b and y with b $\neq 1$, we define the **logarithm of y with base b** as follows: $\log_b y = x$ if and only if $b^x = y$

LAWS OF LOGARITHMS (for M, N, b $> 0,\ b \neq 1$)

$(i)\ \log_b MN = \log_b M + \log_b N$

$(ii)\ \log_b \dfrac{M}{N} = \log_b M - \log_b N$

$(iii)\ \log_b M = \log_b N$ *if and only if* $M = N$

$(iv)\ \log_b M^k = k \cdot \log_b M$

$(v)\ b^{\log_b M} = M$

The logarithmic and exponential functions are **inverse** functions.

Example: Consider $f(x) = 2^x$ and $g(x) = \log_2 x$. Verify that (3,8) is on the graph of f and (8,3) on the graph of g. In addition, $f(g(x)) = 2^{\log_2 x} = x$ *and* $g(f(x)) = \log_2(2^x) = x$ which verifies that f and g are indeed inverse functions.

Simplify.

27) $\log_5 \dfrac{1}{125} =$

28) $\log_4 \sqrt[4]{4} =$

29) $\log_{\sqrt{5}} 125 =$

30) $\frac{1}{t} \ln e^t$

31) $e^{3\ln(x+5)}$

32) e^{-lnx}

33) $4\ln e^{x^2}$

34) e^{3x+ln5}

35) $e^{3(\ln(x)+\ln 5)}$

Write the given expression as a rational number or as a single logarithm.

36) 5 ln (x) – 3 ln (y) + 2 ln (4x) – 6 ln (y^2)

Limits: $\lim_{x\to c} f(x) = N$ "Limit as x approaches c of f(x) equals N"
LIMIT ≠ CONTINUITY
Understanding the difference between limits and continuity: Limit is the value (y) that the function APPROACHES as you get close to c (either from one side or from both sides). Continuity implies that the limit exists (from both sides) and that the limit = the value at c!

Limit Exists: $\mathbf{lim}_{x\to c^+} \boldsymbol{f(x)} = \mathbf{lim}_{x\to c^-} \boldsymbol{f(x)}$
Continuous: $\mathbf{lim}_{x\to c^+} \boldsymbol{f(x)} = \mathbf{lim}_{x\to c^-} \boldsymbol{f(x)}\ \boldsymbol{AND} = \boldsymbol{f(c)} = \boldsymbol{A}\ (\boldsymbol{where\ A\ is\ a\ constant\ not}\ \pm\infty)$

One-Sided limit: exists if graph approaches a value from one side: + (right) or – (left)
Evaluating Limits:

1. Tables: either from table or by graphing and viewing table looking for values of y approaching the same number. Check one sided limits first and if they are equal the overall limit exists.
2. Graphically: Check one sided limits (again – value of limit is the y-value or the height of the graph) and if they are equal overall limit exists.
3. Algebraically: plug in c. If you get a constant then limit exists. Rational functions may need to be simplified first!

Properties:

1. $\lim_{x\to c} A = A$: the limit of a constant A is the constant A- think of it as the graph of a horizontal line. Y = A independent of what the x value is.
2. $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$: you can separate sum and difference of terms.
3. $\lim_{x\to c} A(f(x)) = A \lim_{x\to c} f(x)$: you can pull out numeric factors.
4. $\lim_{x\to c}(f(x))^n = [\lim_{x\to c} f(x)]^n$ you can evaluate limit first, then raise your solution to the power.
5. $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ you can evaluate the limit of the numerator and denominator separately then evaluate their quotient.

Examples: Evaluate the limits:

37) $\lim_{x\to 3}(2x^2)$

38) $\lim_{x\to -2^+} \frac{1}{x^2-4}$

39) $\lim_{x\to \frac{\pi}{2}}(x \sin x)$

40) If $f(x) = \begin{cases} 2x-3, & if\ x < -1 \\ 2, & if\ x = -1 \\ 5x, & if\ x > -1 \end{cases}$, determine the following:

a. $\lim_{x\to -1^-} f(x)$

b. $\lim_{x\to -1^+} f(x)$

c. $\lim_{x\to -1} f(x)$

d. Is f continuous?

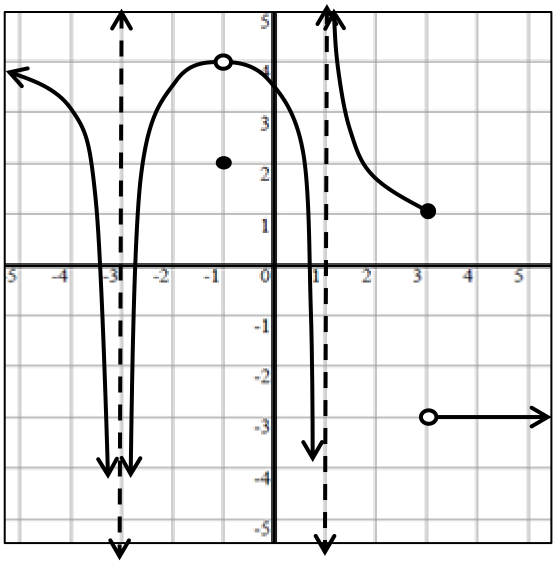
41) If $\lim_{x\to c} f(x) = -2$ and $\lim_{x\to c} g(x) = 5$, determine the following:

a. $\lim_{x\to c} 5f(x)$

b. $\lim_{x\to c}[f(x) - 3g(x)]$

c. $\lim_{x\to c} \frac{f(x)^3}{\sqrt[3]{g(x)}}$

42) Let f(x) be the graph below. Determine the following:



a. $\lim_{x\to -3^-} f(x)$

b. $\lim_{x\to -3^+} f(x)$

c. $\lim_{x\to -3} f(x)$

d. f(-3)

e. Is f continuous at -3?

f. $\lim_{x\to -1^-} f(x)$

g. $\lim_{x\to -1^+} f(x)$

h. $\lim_{x\to -1} f(x)$

i. f(-1)

j. Is f continuous -1?

k. $\lim_{x\to 1^-} f(x)$

l. $\lim_{x\to 1^+} f(x)$

m. $\lim_{x\to 1} f(x)$

n. f(1)

o. Is f continuous at 1?

p. $\lim_{x\to 3^-} f(x)$

q. $\lim_{x\to 3^+} f(x)$

r. $\lim_{x\to 3} f(x)$

s. f(3)

t. Is f continuous at 1?

u. $\lim_{x\to -\infty} f(x)$

v. $\lim_{x\to \infty} f(x)$

Secant and Tangent Lines:

Average Rate of Change: slope through 2 points.

Secant Line: line connecting 2 points on $f(x)$.

$$m_{sec} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Instantaneous Rate of Change: slope at a single point on $f(x)$.

Tangent line: line through a single point with slope approximated at that point.

$$m_{tangent} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

43) Find the instantaneous rate of change of y = $4x^2$ at x = 2

44) Find the average rate of change of y = $4x^2$ between x = 0 and x = 2

45) Determine the slope of the secant line on the curve $f(x) = x^2 + 1$ at x = 1 and x = 3 and the slope of the tangent line at x = 1. Compare your slopes by sketching $f(x)$, the secant line through x = 1 and x =3 and the tangent line at x = 1.

46) Determine the slope of the secant line on the curve $f(x) = 3x^2$ at x = 1 and x = 2 and the slope of the tangent line at x = 1.5. Compare your slopes by sketching $f(x)$, the secant line through x = 1 and x =2 and the tangent line at x = 1.5.

ANSWER KEY

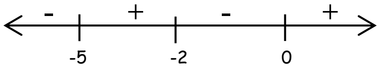
1. $(3\text{x} + 5)(\text{x} - 2)$

2. $(5x + 4y)(5x - 4y)$

3. $(x - 3)(y - 2)$

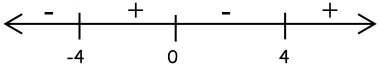
4. $\frac{2x^{\left(\frac{1}{2}\right)}(x+4)(x+1)}{(x+2)}$

5. Sign Chart:



Solution: $(-5,-2) \cup (0,\infty)$

6. Sign Chart:



Solution: $(-\infty,-4] \cup [0,4]$

7. $f(x) = 2(x-5)^2 + 7$
Domain: $(-\infty,\infty)$
Range: $[7,\infty)$
x-intercept(s): none
y-intercept(s): $(0,57)$
Extrema (by hand): absolute minimum: $(5,7)$
End Behavior:
$\lim_{x\to-\infty} f(x) = \infty$
$\lim_{x\to\infty} f(x) = \infty$
Interval of Continuity: $(-\infty,\infty)$
Tests for Symmetry:
$f(-x) = 2x^2 + 20x + 57\, neither$

8. $g(x) = 3x(x^2 - 2)$
Domain: $(-\infty,\infty)$
Range: $(-\infty,\infty)$
x-intercept(s): $(0,0),(\sqrt{2},0),(-\sqrt{2},0)$
y-intercept(s): $(0,0)$
Extrema (on calc): local max: 3.266 local min: -3.266
End Behavior:
$\lim_{x\to-\infty} g(x) = -\infty$
$\lim_{x\to\infty} g(x) = \infty$
Interval of Continuity: $(-\infty,\infty)$
Tests for Symmetry:
$g(-x) = -3x^3 + 6x = -g(x) ODD$

9. Domain: $(-\infty,\infty)$
Range: $(0,4]$
x-intercept(s): none
y-intercept(s): $(0,4)$
End Behavior:
$\lim_{x\to-\infty} \frac{8}{2+x^2} = 0$
$\lim_{x\to\infty} \frac{8}{2+x^2} = 0$
Interval of Continuity: $(-\infty,\infty)$
Tests for Symmetry:
$f(-x) = \frac{8}{2+x^2} = f(x) EVEN$

10. $h(x) = \frac{x+1}{x-1}$

11. $\frac{5x}{(x-3)(x-2)(x+2)}$

12. $\frac{1}{(x+y)(x^2+y^2)}$

13. $\frac{x^3+2x^2+9x+8}{4(x+2)^2(x^2+1)}$

14. -1

15. 3

16. -1

17. 3

18. 2

19. 1

20. ½

21. - ½

22. $-\frac{\sqrt{2}}{2}$

23. $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

24. $x = \frac{\pi}{4}, \frac{5\pi}{4}$

25. $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

26. $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

27. -3

28. ¼

29. 6

30. 1

31. $(x+5)^3$

32. $\frac{1}{x}$

33. $4x^2$

34. $5e^{3x}$

35. $125x^3$

36. $\ln \frac{16x^7}{y^{15}}$

37. 18

38. -∞

39. $\frac{\pi}{2}$

40.
a. -5
b. -5
c. -5
d. NO

41.
a. -10
b. -17
c. $-\frac{8}{\sqrt[3]{5}}$

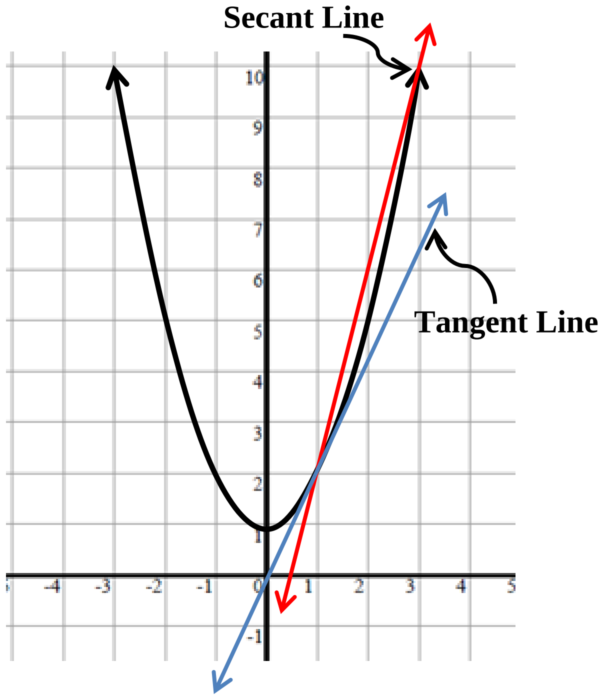
42.
a. -∞
b. -∞
c. -∞
d. DNE
e. NO
f. 4
g. 4
h. 4
i. 2
j. NO
k. -∞
l. ∞
m. DNE
n. DNE
o. NO
p. 1
q. -3
r. DNE
s. 1
t. NO
u. 4
v. -3

43. $\lim_{h\to 0} \frac{4(x+h)^2-4x^2}{h} =$
$\lim_{h\to 0} \frac{4x^2+8xh+4h^2-4x^2}{h} = \lim_{h\to 0} \frac{8xh+4h^2}{h} = 8x$
At x = 2: $8(2) = 16$

44. $\frac{f(2)-f(0)}{2-0} = \frac{4(2^2)-4(0^2)}{2} = 8$

45. $m_{sec} = \frac{f(3)-f(1)}{3-1} = 4$

$m_{tangent} = \lim_{h\to 0} \frac{(x+h)^2+1-(x^2+1)}{h} = 2x$
$m_{tangent}\ at\ x = 1\ \ is\ 2$



46. $m_{sec} = \frac{f(2)-f(1)}{2-1} = 9$

$m_{tangent} = \lim_{h\to 0} \frac{3(x+h)^2-3x^2}{h} = 6x$
$m_{tangent}\ at\ x = 1.5\ \ is\ 9$