

AP Calculus Summer Prep

Topics from Algebra and Pre-Calculus

(Solutions are the last pages of this handout)

The purpose of this packet is to give you a review of basic skills. In calculus you will need to recall every math concept you have ever learned 😊, but this packet will help you recall those topics that are most important and that will show up in the beginning of the year.

You are asked to have this packet completed for the first day of school and **all problems are expected to be completed without the use of a graphing calculator.** Questions will be answered on the first day of class.

ALGEBRA TOPICS

FACTORIZING POLYNOMIALS AND RATIONALS: basic factoring and factoring of binomials, fractional exponents and binomials within rational expressions to simplify.

Factor completely:

1) $x^3(x+4)^5 - 2x^2(x+4)^6$

2) $2x^{\left(\frac{1}{2}\right)}(x+2)^2 + 2x^{\left(\frac{3}{2}\right)}(x+2)^3$

3) $\frac{8x(x+5)^3 - 4x^2(x+5)^2}{16x^2 + 80x}$

4) $\frac{3(2x^2-8)+12(x+2)^2}{12x^2+6x-36}$

SOLVING POLYNOMIAL INEQUALITIES

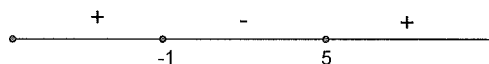
Example 1:

$$x^2 - 4x - 5 < 0$$

Solution:

$$(x+1)(x-5) < 0$$

factor and determine critical values: $x = -1, 5$



Answer: $(-1, 5)$

*mark the zeros; pick a test point to determine the sign of the polynomial in each interval- this is called a **SIGN CHART!***

PRACTICE PROBLEMS - Include a Sign Chart

5) $x^3 + 7x^2 + 10x > 0$

6) $3x(x+2)^3 + 9x^2(x+2)^2 \leq 0$

PRE-CALCULUS TOPICS

Analyzing Graphs:

Domain and Range, x-intercepts (zeros) and y-intercepts, extrema (local and absolute)

End Behavior (limits at infinity): $\lim_{x \rightarrow \pm\infty} f(x)$

Continuity and Discontinuity: All Polynomials are continuous for all x .

Types of discontinuities: removable (graph as hole), removable with point (piece-wise), infinite (vertical asymptote) and jump (piece-wise).

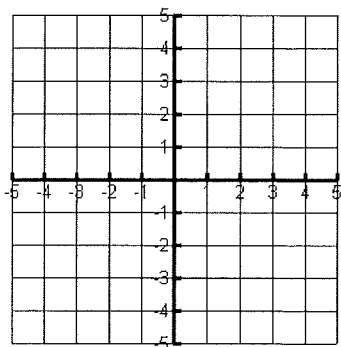
Even/Odd Functions:

Even: Symmetric to the y-axis. Algebraically: $f(-x) = f(x)$

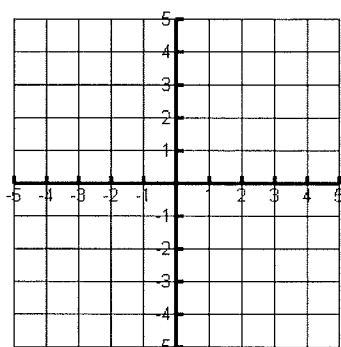
Odd: Symmetric to the origin. Algebraically: $f(-x) = -f(x)$

Parent Functions that all graduating Pre-Calculus students should be able to sketch without the use of a calculator. Determine any key characteristics, domain, interval of continuity, symmetry, end behavior and type and location of any discontinuities.

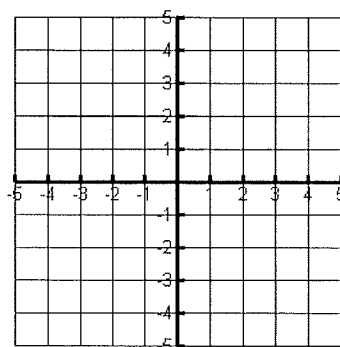
Linear: $y = x$



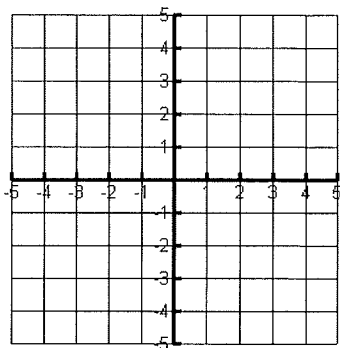
Quadratic: $y = x^2$



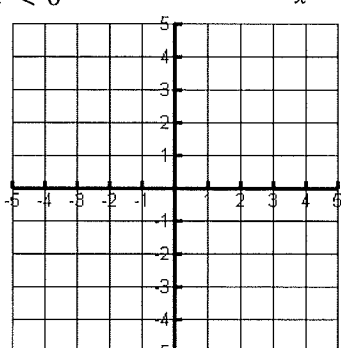
Cubic: $y = x^3$



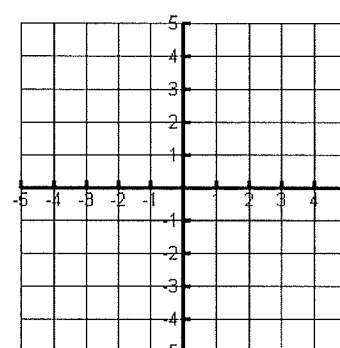
Absolute Value: $y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$



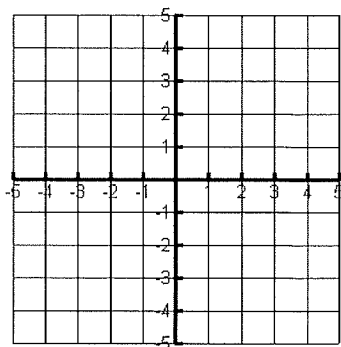
Rational: $y = \frac{1}{x}$



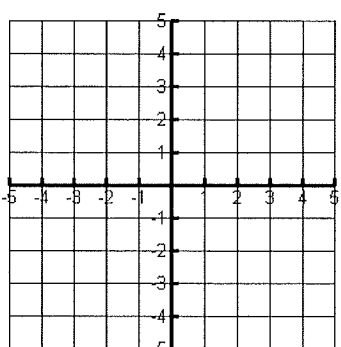
Root: $y = \sqrt{x}$



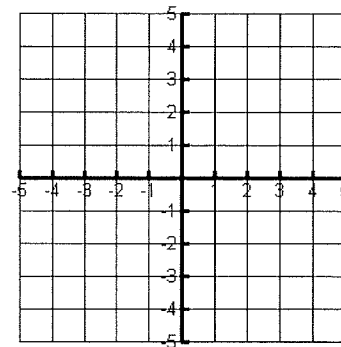
Exponential Growth: $y = e^x$
where $x > 0$



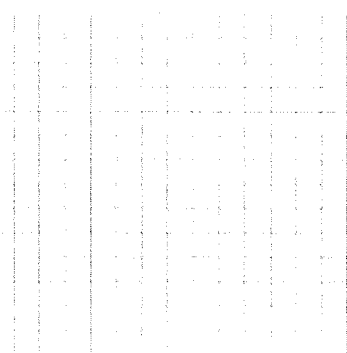
Exponential Decay: $y = e^x$
where $x < 0$



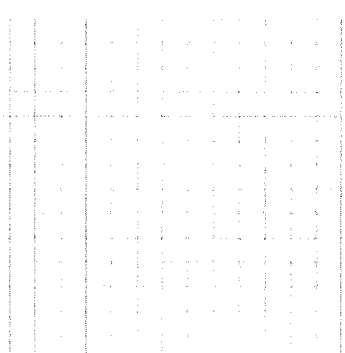
Logarithmic Growth: $y = \ln x$



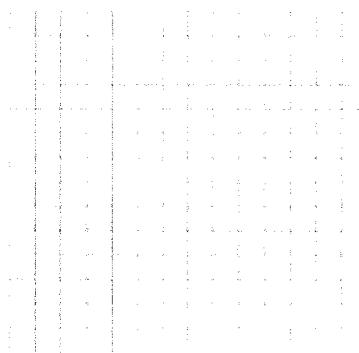
Sine: $y = \sin(x)$



Cosine: $y = \cos(x)$



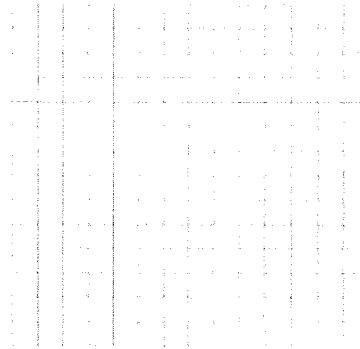
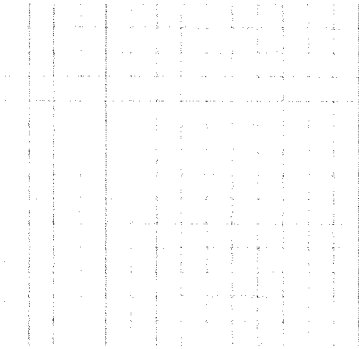
Tangent: $y = \tan(x)$



7 - 8 Write the absolute value function as a piecewise function then identify the vertex and sketch the graph.

7) $f(x) = 2|x - 3| + 1$

8) $f(x) = |1 - x|$



Analyze the following functions without graphing them on a graphing calculator (except where noted).

9) $f(x) = x^2 - 2x - 3$

10) $g(x) = 6(x + 2)^2 - 4(x + 2)^3$

Domain:

Domain:

Range:

Range:

x-intercept(s):

x-intercept(s):

y-intercept(s):

y-intercept(s):

Extrema (by hand):

End Behavior:

End Behavior:

$$\lim_{x \rightarrow -\infty} f(x) =$$

$$\lim_{x \rightarrow -\infty} g(x) =$$

$$\lim_{x \rightarrow \infty} f(x) =$$

$$\lim_{x \rightarrow \infty} g(x) =$$

Interval of Continuity:

Interval of Continuity:

Tests for Symmetry:

Rational Expressions and Functions:

Domain: excludes all zeros of the denominator - both removable and non-removable.

Range: effected by any horizontal asymptotes. Recall, a function may cross a horizontal asymptote.

End Behavior (limits at infinity): $\lim_{x \rightarrow \pm\infty} f(x) = \text{horizontal and Oblique asymptote}$

Discontinuity: occur at the zeros of the denominator:

1. Zeros of the denominator that **CANNOT** be simplified: infinite discontinuities graph as vertical Asymptotes.

2. Zeros of the denominator that **CAN** be simplified: removable discontinuities graph as holes.

Simplifying: factor out and cancel common factors, simplify complex fractions by multiplying through by the LCD over itself.

Solving: relate terms to zero, combine all terms, determine the zeros of the numerator and denominator (called critical values) and create a sign chart to determine the signs before, between and after all critical values.

11) Determine the following for the function $f(x) = \frac{3x^2+2x-1}{2x^2+x-1}$

Domain:

Range:

x-intercept(s):

y-intercept(s):

End Behavior:

$$\lim_{x \rightarrow -\infty} f(x) =$$

$$\lim_{x \rightarrow \infty} f(x) =$$

Interval of Continuity:

Name and Location of Discontinuities:

12) Simplify

a. $\frac{3e^x x^2 - 3e^x}{e^{4x}(x+1)^3}$

b. $\frac{3x^{-2}}{1x^{-1} - \frac{1}{3}x^{-4}}$

c. $\frac{\frac{1}{x+1}}{\frac{2}{x-3} - \frac{1}{x^2-2x-3}}$

13) Solve

a. $\frac{x+3}{x-1} \leq 4$

b. $\frac{5x(x+1)+10(x+1)^2}{20(x+2)(x+1)^3} \leq 0$

FUNCTION OPERATIONS AND COMPOSITION; INVERSE FUNCTIONS

Composition of a function g with a function f is defined as:

$$h(x) = g(f(x))$$

The domain of h is the set of all x -values such that x is in the domain of f and $f(x)$ is in the domain of g .

Inverses: Functions f and g are inverses of each other provided:

$$f(g(x)) = x \text{ and } g(f(x)) = x$$

The function g is denoted as f^{-1} , read as "f inverse" and $f^{-1}(x) \neq \frac{1}{f(x)}$

Horizontal Line Test: If any horizontal line which is drawn through the graph of a function f intersects the graph no more than once, then f is said to be a **one-to-one** function and has an inverse.

14) Let f and g be functions whose values are given by the table below. Assume g is one-to-one.

a. $f(g(3))$

b. $g^{-1}(4)$

c. $f(g^{-1}(6))$

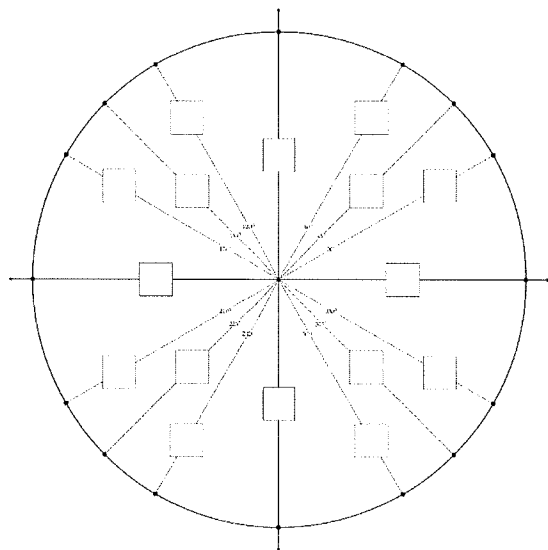
d. $f^{-1}(f(g(2)))$

x	$f(x)$	$g(x)$
1	6	2
2	9	3
3	10	4
4	-1	6

TRIGONOMETRY

UNIT CIRCLE: Complete angles and Sine/Cosine

Key Trig Identities - formulas you should know:



$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \sin(2x) &= 2 \sin x \cos x \\ \cos(2x) &= 1 - 2 \sin^2 x \\ 1 + \tan^2 x &= \sec^2 x \\ \sin(-x) &= -\sin x \text{ ODD} \\ \cos(-x) &= \cos x \text{ EVEN} \end{aligned}$$

15) Without using a calculator or table, find each value:

a. $\cos\left(-\frac{\pi}{3}\right)$

b. $\sec\left(\frac{11\pi}{6}\right)$

c. $\tan^{-1}(-1)$

d. $\sec^{-1}(-2)$

16) Solve the trigonometric equations algebraically by using identities and **without** the use of a calculator. Find all solutions in the interval $0 \leq \theta \leq 2\pi$.

a. $2 \sin^2 \theta - 1 = 0$

b. $\sin(2x) = \cos x$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS:

Logarithm: For any positive numbers b and y with $b \neq 1$, we define the **logarithm of y with base b** as follows: $\log_b y = x$ if and only if $b^x = y$

LAWS OF LOGARITHMS (for $M, N, b > 0, b \neq 1$)

(i) $\log_b MN = \log_b M + \log_b N$

(ii) $\log_b \frac{M}{N} = \log_b M - \log_b N$

(iii) $\log_b M = \log_b N$ if and only if $M = N$

(iv) $\log_b M^k = k \cdot \log_b M$

(v) $b^{\log_b M} = M$

The logarithmic and exponential functions are **inverse** functions.

Example: Consider $f(x) = 2^x$ and $g(x) = \log_2 x$. Verify that (3,8) is on the graph of f and (8,3) on the graph of g . In addition, $f(g(x)) = 2^{\log_2 x} = x$ and $g(f(x)) = \log_2(2^x) = x$ which verifies that f and g are indeed inverse functions.

Simplify.

17) $\log_5 \frac{1}{125}$

18) $\log_4 \sqrt[8]{16}$

19) $2 \ln \frac{3}{e^3}$

20) $\frac{1}{t} \ln e^t$

21) $e^{3 \ln(x+5)}$

22) $e^{-\ln x}$

23) $4 \ln e^{x^2}$

24) $e^{3x + \ln 5}$

25) $e^{3(\ln(x) + \ln 5)}$

26) Write $5 \ln(x) - 3 \ln(y) + 2 \ln(4x) - 6 \ln(y^2)$ as a single logarithm.

27) Expand $\ln\left(\frac{3x^2}{2\sqrt{y^2-4}}\right)$.

28) Evaluate:

a. $\ln 1$

b. $\ln 3e$

c. $\ln e$

d. $\ln 0$

e. $\ln(\ln e)$

Limits: $\lim_{x \rightarrow c} f(x) = N$ "Limit as x approaches c of $f(x)$ equals N "

LIMIT \neq CONTINUITY

Understanding the difference between limits and continuity: Limit is the value (y) that the function APPROACHES as you get close to c (either from one side or from both sides). Continuity implies that the limit exists (from both sides) and that the limit = the value at c !

Limit Exists: $\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) \therefore \lim_{x \rightarrow c} f(x)$ exists and will equal all the same value or $\pm\infty$.

Continuous: $\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) = f(c) = A$ (where A is a constant not $\pm\infty$) **MUST SHOW ALL 3!!**

One-Sided limit: exists if graph approaches a value from one side: + (right) or - (left)

$$\lim_{x \rightarrow a} C = C$$

Evaluating Limits:

- Tables:** either from table or by graphing and viewing table looking for values of y approaching the same number. Check one sided limits first and if they are equal the overall limit exists.
- Graphically:** Check one sided limits (again - value of limit is the y -value or the height of the graph) and if they are equal overall limit exists.
- Algebraically - Direct substitution:** plug in c . If you get a constant then limit exists. Rational functions may need to be simplified first!

When you use Direct substitution and you get

- $\frac{0}{\text{number}}$ then the limit is equal to 0
- $\frac{0}{0}$ most likely a removable discontinuity, try simplifying or rationalizing.
- $\frac{\text{number}}{0}$ most likely an infinite discontinuity, look at each side and evaluate one sided limit or if asking for two sided limit make sure the function is approaching the same from both sides.

$\lim_{x \rightarrow \pm\infty} f(x)$: End Behavior:

- Polynomial: use degree (even or odd) and sign of leading coefficient to decide parabolic or cubic.
- Rational: end behavior is determined by the Horizontal or Oblique Asymptotes.
- Other "Known" Graphs: know the graphs!

Examples: Evaluate the limits:

29) $\lim_{x \rightarrow 3} (2x^2)$

30) $\lim_{x \rightarrow -2^+} \frac{1}{x+2}$

31) $\lim_{x \rightarrow \frac{\pi}{2}} (x \sin x)$

32) $\lim_{x \rightarrow 5} \frac{3}{x-5}$

33) $\lim_{x \rightarrow 0^+} \ln(x)$

34) $\lim_{x \rightarrow -\frac{\pi}{2}} \tan(x)$

35) $\lim_{x \rightarrow -\infty} 3e^{2x}$

36) $\lim_{x \rightarrow 1} \frac{3x^2 - 2x - 1}{x^2 - 1}$

37) $\lim_{x \rightarrow \infty} \frac{2x}{x-1}$

38) $\lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x-1}$

39) $\lim_{x \rightarrow -\infty} 2x(x-3)^2(x+1)$

40) $\lim_{x \rightarrow -\infty} \frac{3}{x-5}$

41) If $f(x) = \begin{cases} 2x - 3, & \text{if } x < -1 \\ 2, & \text{if } x = -1 \\ 5x, & \text{if } x > -1 \end{cases}$, determine the following:

a. $\lim_{x \rightarrow -1^-} f(x)$

b. $\lim_{x \rightarrow -1^+} f(x)$

c. $\lim_{x \rightarrow -1} f(x)$

d. Is f continuous?

42) If $f(x) = |x + 2| = \begin{cases} x + 2, & x \geq -2 \\ -(x + 2), & x < -2 \end{cases}$, determine the following:

a. $\lim_{x \rightarrow -2^-} f(x)$

b. $\lim_{x \rightarrow -2^+} f(x)$

c. $\lim_{x \rightarrow -2} f(x)$

d. Is f continuous?

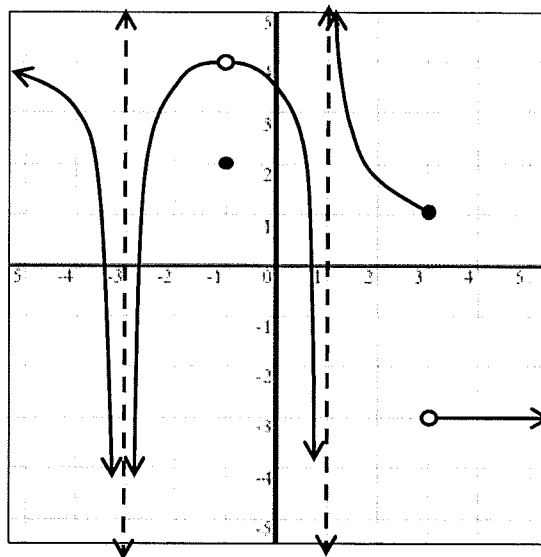
43) Determine whether $f(x)$ is continuous. Justify. Then, determine domain, interval of continuity and location and type of any discontinuities.

a. $f(x) = \begin{cases} 3x^2, & x < 0 \\ 4, & x = 0 \\ 3 \sin^2 x, & x > 0 \end{cases}$

b. $f(x) = \begin{cases} \ln x, & x < e^2 \\ \sqrt{x}, & x \geq e^2 \end{cases}$

c. $f(x) = \begin{cases} 3x^2 - 1, & x \leq 2 \\ 5x + 1, & x > 2 \end{cases}$

44) Let $f(x)$ be the graph below. Determine the following:



a. $\lim_{x \rightarrow -3^-} f(x) =$

b. $\lim_{x \rightarrow -3^+} f(x) =$

c. $\lim_{x \rightarrow -3} f(x) =$

d. $f(-3) =$

e. Is f continuous at -3 ?

f. $\lim_{x \rightarrow -1^-} f(x) =$

g. $\lim_{x \rightarrow -1^+} f(x) =$

h. $\lim_{x \rightarrow -1} f(x) =$

i. $f(-1) =$

j. $\lim_{x \rightarrow 1^-} f(x) =$

k. $\lim_{x \rightarrow 1^+} f(x) =$

l. $\lim_{x \rightarrow 1} f(x) =$

m. $f(1) =$

n. $\lim_{x \rightarrow 3^-} f(x) =$

o. $\lim_{x \rightarrow 3^+} f(x) =$

p. $\lim_{x \rightarrow 3} f(x) =$

q. $f(3) =$

r. $\lim_{x \rightarrow -\infty} f(x) =$

s. $\lim_{x \rightarrow \infty} f(x) =$

t. Continuity Interval:

u. Location and type of each discontinuity:

v. Domain:

45) Draw a graph given the following conditions:

◆ $f(0) = 0$

◆ $f(1) = 2$

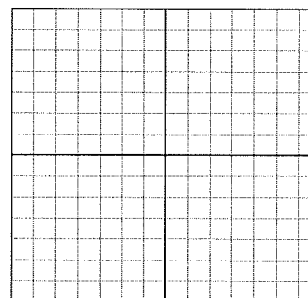
◆ $f(4) = -3$

◆ $\lim_{x \rightarrow 3^-} f(x) = \infty$

◆ $\lim_{x \rightarrow -\infty} f(x) = -1$

◆ $\lim_{x \rightarrow \infty} f(x) = 1$

◆ $\lim_{x \rightarrow 3^+} f(x) = -\infty$



46) Draw a graph given the following conditions:

◆ $f(-2) = 0$

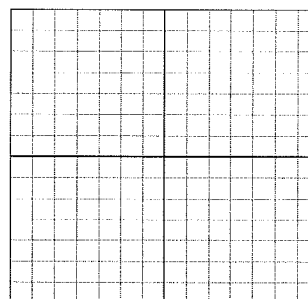
◆ $f(2) = 2$

◆ $\lim_{x \rightarrow -2^-} f(x) = 4$

◆ $\lim_{x \rightarrow -\infty} f(x) = -\infty$

◆ $\lim_{x \rightarrow \infty} f(x) = 5$

◆ $\lim_{x \rightarrow -2^+} f(x) = 0$



ALGEBRA TOPICS

FACTORING POLYNOMIALS AND RATIONALS: basic factoring and factoring of binomials, fractional exponents and binomials within rational expressions to simplify.

Factor completely:

1) $x^3(x+4)^5 - 2x^2(x+4)^6$

$$x^2(x+4)^5 [x - 2(x+4)]$$

$$x^2(x+4)^5 (-x-8)$$

OR $-x^2(x+4)^5(x+8)$

2) $2x^{(\frac{1}{2})}(x+2)^2 + 2x^{(\frac{3}{2})}(x+2)^3$

$$2x^{1/2}(x+2)^2 [1 + x(x+2)]$$

$$2x^{1/2}(x+2)^2 (1+x^2+2x)$$

$$2x^{1/2}(x+2)^2 (x+1)^2$$

3) $\frac{8x(x+5)^3 - 4x^2(x+5)^2}{16x^2 + 80x}$

$$\frac{4x(x+5)^2 [2(x+5) - x]}{4 \cdot 16x(x+5)}$$

$$\frac{(x+5)(x+10)}{4} \text{ OR } \frac{1}{4}(x+5)(x+10)$$

4) $\frac{3(2x^2-8)+12(x+2)^2}{12x^2+6x-36}$

$$\frac{6(x^2-4)+12(x+2)^2}{6(2x^2+x-6)} = \frac{6(x+2)[(x-2)+2(x+2)]}{6(2x-3)(x+2)}$$

$$= \frac{3x+2}{2x-3}$$

SOLVING POLYNOMIAL INEQUALITIES

Example 1: $x^2 - 4x - 5 < 0$

Solution: $(x+1)(x-5) < 0$

factor and determine critical values: $x = -1, 5$



Answer: $(-1, 5)$

mark the zeros; pick a test point to determine the sign of the polynomial in each interval- this is called a **SIGN CHART!**

PRACTICE PROBLEMS - Include a Sign Chart

5) $x^3 + 7x^2 + 10x > 0$

$$x(x^2 + 7x + 10) > 0$$

$$x(x+5)(x+2) > 0$$

C.V. $x = 0, -5, -2$

Solution: $(-5, -2) \cup (0, \infty)$

6) $3x(x+2)^3 + 9x^2(x+2)^2 \leq 0$

$$3x(x+2)^2 [(x+2) + 3x] \leq 0$$

$$3x(x+2)^2 (4x+2) \leq 0 \text{ OR } 6x(x+2)^2 (2x+1) \leq 0$$

C.V. $x = 0, -2, -1/2$

Solution: $x = -2$ and $[-1/2, 0]$

PRE-CALCULUS TOPICS

Analyzing Graphs:

Domain and Range, x-intercepts (zeros) and y-intercepts, extrema (local and absolute)

End Behavior (limits at infinity): $\lim_{x \rightarrow \pm\infty} f(x)$

Continuity and Discontinuity: All Polynomials are continuous for all x .

Types of discontinuities: removable (graph as hole), removable with point (piece-wise), infinite (vertical asymptote) and jump (piece-wise).

Even/Odd Functions:

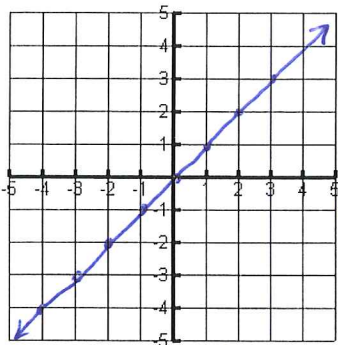
Even: Symmetric to the y-axis. Algebraically: $f(-x) = f(x)$

Odd: Symmetric to the origin. Algebraically: $f(-x) = -f(x)$

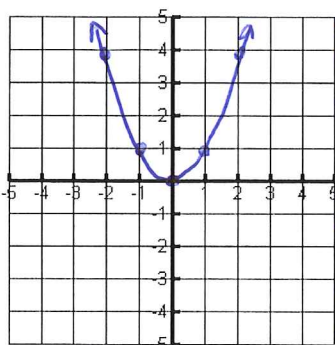
Parent Functions that all graduating Pre-Calculus students should be able to sketch without the use of a calculator. Determine any key characteristics, domain, interval of continuity, (i.o.c.) symmetry, end behavior and type and location of any discontinuities.

Linear: $y = x$ $m=1$ $b=1$

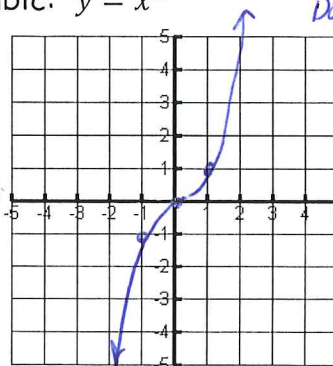
Quadratic: $y = x^2$ vertex: $(0,0)$ Cubic: $y = x^3$



Domain: $(-\infty, \infty)$
i.o.c.: $(-\infty, \infty)$
Symmetry: $f(-x) = -x$ odd
No discont.



Domain: $(-\infty, \infty)$
i.o.c.: $(-\infty, \infty)$
Symmetry: $f(-x) = x^2$ even
No discont.



Domain: $(-\infty, \infty)$
i.o.c.: $(-\infty, \infty)$
Symmetry: $f(-x) = -x^3$ odd
No discont.

E.B: $\lim_{x \rightarrow -\infty} y = -\infty$ $\lim_{x \rightarrow \infty} y = \infty$

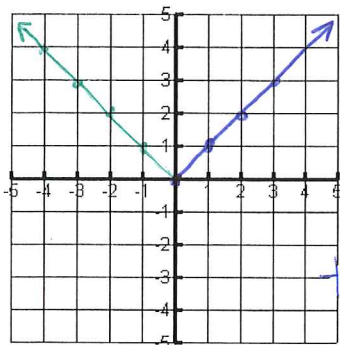
E.B: $\lim_{x \rightarrow -\infty} y = \infty$ $\lim_{x \rightarrow \infty} y = \infty$

E.B: $\lim_{x \rightarrow -\infty} y = -\infty$ $\lim_{x \rightarrow \infty} y = \infty$

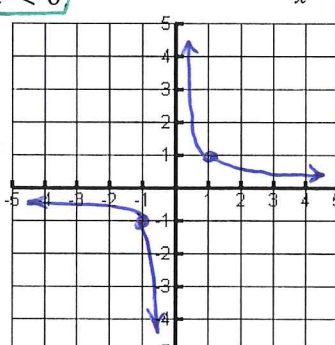
Absolute Value: $y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Rational: $y = \frac{1}{x}$

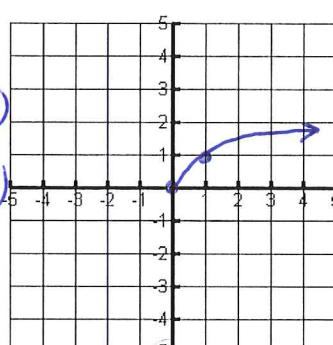
Root: $y = \sqrt{x}$



vertex: $(0,0)$
Domain: $(-\infty, \infty)$
i.o.c.: $(-\infty, \infty)$
Symmetry: $f(-x) = |-x| = |x|$ even
no discont.



VA: $x=0$
HA: $y=0$
Domain: $(-\infty, 0) \cup (0, \infty)$
i.o.c.: $(-\infty, 0) \cup (0, \infty)$
 $f(-x) = -\frac{1}{x}$
Symmetry: odd



Domain: $[0, \infty)$
i.o.c.: $(0, \infty)$
 $x=0$ is cont. from the right
Symmetry: $f(-x) = \sqrt{-x}$ no symmetry
no point+
disc.

E.B: $\lim_{x \rightarrow -\infty} y = \infty$ $\lim_{x \rightarrow \infty} y = \infty$

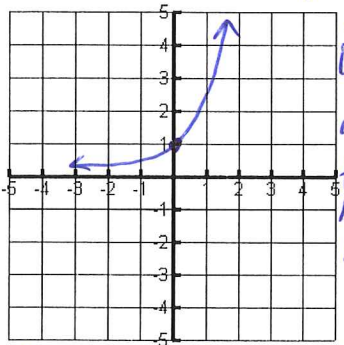
E.B: $\lim_{x \rightarrow -\infty} y = \lim_{x \rightarrow \infty} y = 0$

E.B: $\lim_{x \rightarrow -\infty} y = \text{DNE}$ $\lim_{x \rightarrow \infty} y = \infty$

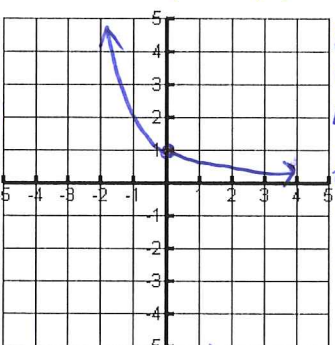
Exponential Growth: $y = e^x$ where $x > 0$ pt. $(0,1)$

Exponential Decay: $y = e^{-x}$ where $x < 0$ pt. $(0,1)$ HA: $y=0$

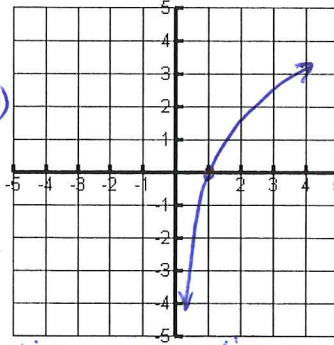
Logarithmic Growth: $y = \ln x$ pt. $(1,0)$ VA: $x=0$



HA: $y=0$
Domain: $(-\infty, \infty)$
i.o.c.: $(-\infty, \infty)$
 $f(-x) = e^{-x}$
no symmetry
no discont.



Domain: $(-\infty, \infty)$
i.o.c.: $(-\infty, \infty)$
 $f(-x) = e^x$
no symm.
no discont.



Domain: $(0, \infty)$
i.o.c.: $(0, \infty)$
Symmetry: $f(-x) = \ln(-x)$ no symm.
no pt. discontinuit

E.B: $\lim_{x \rightarrow -\infty} y = 0$ $\lim_{x \rightarrow \infty} y = \infty$

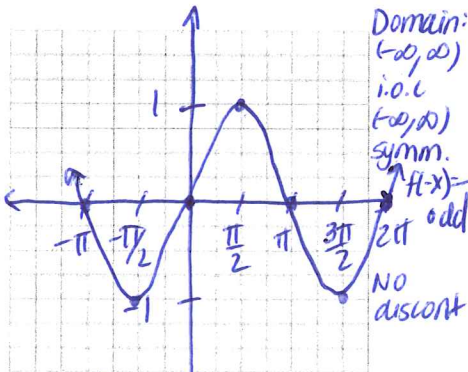
E.B: $\lim_{x \rightarrow -\infty} y = \infty$ $\lim_{x \rightarrow \infty} y = 0$

E.B: $\lim_{x \rightarrow -\infty} y = \text{DNE}$ $\lim_{x \rightarrow \infty} y = \infty$

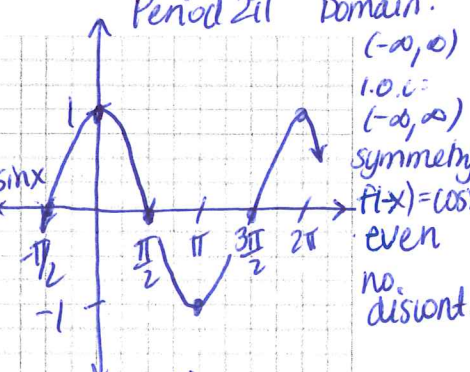
Sine: $y = \sin(x)$ period: 2π

Cosine: $y = \cos(x)$

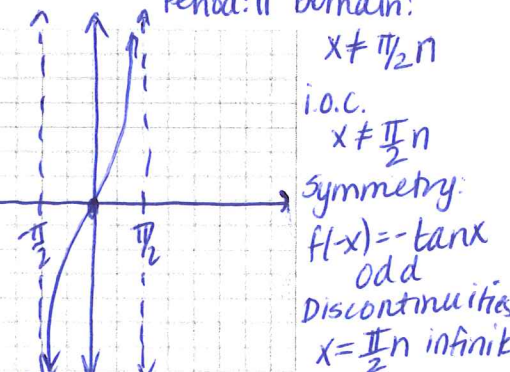
Tangent: $y = \tan(x)$



Domain: $(-\infty, \infty)$
i.o.c.: $(-\infty, \infty)$
Symm. $f(-x) = \sin(-x)$ odd
No discont



Period 2π Domain: $(-\infty, \infty)$
i.o.c.: $(-\infty, \infty)$
Symmetry: $f(-x) = \cos(x)$ even
no discont.



Period: π Domain: $x \neq \pi/2 + n\pi$
i.o.c.: $x \neq \pi/2 + n\pi$
Symmetry: $f(-x) = -\tan(x)$ odd
Discontinuities $x = \pi/2 + n\pi$ infinite

$\lim_{x \rightarrow -\infty} y = \lim_{x \rightarrow \infty} y = \text{DNE}$

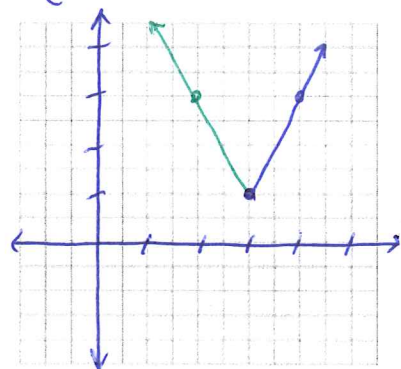
E.B: $\lim_{x \rightarrow -\infty} y = \lim_{x \rightarrow \infty} y = \text{DNE}$

E.B: $\lim_{x \rightarrow -\infty} y = \lim_{x \rightarrow \infty} y = \text{DNE}$

7 - 8 Write the absolute value function as a piecewise function then identify the vertex and sketch the graph.

7) $f(x) = 2|x - 3| + 1$

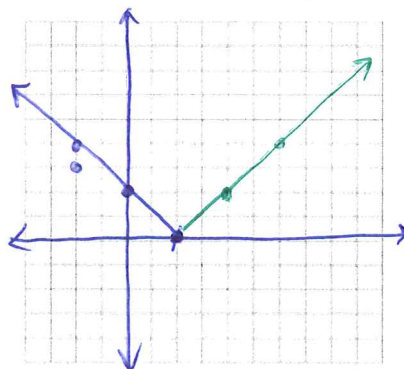
$$f(x) = \begin{cases} 2(x-3)+1, & x \geq 3 \\ -2(x-3)+1, & x < 3 \end{cases} = \begin{cases} 2x-5, & x \geq 3 \\ -2x+7, & x < 3 \end{cases}$$



vertex:
(3, 1)

8) $f(x) = |1 - x|$

$$f(x) = \begin{cases} 1-x, & x \leq 1 \\ -(1-x), & x > 1 \end{cases} = \begin{cases} 1-x, & x \leq 1 \\ -1+x, & x > 1 \end{cases}$$



vertex: (1, 0)

Analyze the following functions without graphing them on a graphing calculator (except where noted).

9) $f(x) = x^2 - 2x - 3 = \frac{(x-3)(x+1)}{(x-1)^2 - 1 - 3} = (x-1)^2 - 4$

Domain: $(-\infty, \infty)$

Range: $[-4, \infty)$

x-intercept(s): $x = 3, -1$

y-intercept(s): $y = -3$

Extrema (by hand): *absolute max = -4*

End Behavior:

$$\lim_{x \rightarrow -\infty} f(x) = \infty$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

Interval of Continuity:

$(-\infty, \infty)$

Tests for Symmetry:

$$f(-x) = (-x)^2 - 2(-x) - 3$$

$$= x^2 + 2x - 3 \text{ none}$$

10) $g(x) = 6(x+2)^2 - 4(x+2)^3 = 2(x+2)^2 [3 - 2(x+2)] = 2(x+2)^2 (-2x-1) = -2(x+2)^2 (2x+1)$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

x-intercept(s): $x = -2, -1/2$

y-intercept(s): $y = -8$

End Behavior: *leading term: $(-2)(x^2)(2x) = -4x^3$* ↑
↓

$$\lim_{x \rightarrow -\infty} g(x) = \infty$$

$$\lim_{x \rightarrow \infty} g(x) = -\infty$$

Interval of Continuity:

$(-\infty, \infty)$

Rational Expressions and Functions:

Domain: excludes all zeros of the denominator - both removable and non-removable.

Range: effected by any horizontal asymptotes. Recall, a function may cross a horizontal asymptote.

End Behavior (limits at infinity): $\lim_{x \rightarrow \pm\infty} f(x) = \text{horizontal and Oblique asymptote}$

Discontinuity: occur at the zeros of the denominator:

1. Zeros of the denominator that CANNOT be simplified: infinite discontinuities graph as vertical Asymptotes.
2. Zeros of the denominator that CAN be simplified: removable discontinuities graph as holes.

Simplifying: factor out and cancel common factors, simplify complex fractions by multiplying through by the LCD over itself.

Solving: relate terms to zero, combine all terms, determine the zeros of the numerator and denominator (called critical values) and create a sign chart to determine the signs before, between and after all critical values.

11) Determine the following for the function $f(x) = \frac{3x^2+2x-1}{2x^2+x-1} = \frac{(3x-1)(x+1)}{(2x-1)(x+1)}$ HA: $y = 3/2$

Domain: $(-\infty, -1) \cup (-1, 1/2) \cup (1/2, \infty)$

Range: $(-\infty, 3/2) \cup (3/2, \infty)$

x-intercept(s): $x = 1/3$ OR $(1/3, 0)$

y-intercept(s): $y = -2$ OR $(0, -2)$

End Behavior:

$$\lim_{x \rightarrow -\infty} f(x) = 3/2$$

$$\lim_{x \rightarrow \infty} f(x) = 3/2$$

Interval of Continuity: $(-\infty, -1) \cup (-1, 1/2) \cup (1/2, \infty)$ Name and Location of Discontinuities:

$x = -1$ removable

$x = 1/2$ infinite

12) Simplify

a. $\frac{3e^x x^2 - 3e^x}{e^{4x}(x+1)^3} = \frac{3e^x(x^2-1)}{(e^x)^4(x+1)^3}$

b. $\frac{3x^{-2}}{1x^{-1} - \frac{1}{3}x^{-4}} = \frac{(\frac{3}{x^2}) \cdot 3x^4}{(\frac{1}{x} - \frac{1}{3x^4}) \cdot 3x^4}$

c. $\frac{\frac{1}{x+1}}{\frac{2}{x-3} - \frac{1}{x^2-2x-3}} = \frac{(x+1)(x-3)}{(x+1)(x-3)}$

$$\frac{3(x+1)(x-1)}{e^{3x}(x+1)^3}$$

$$\frac{9x^2}{3x^3-1}$$

$$\frac{x-3}{2(x+1)-1} = \frac{x-3}{2x+1}$$

$$= \frac{3(x-1)}{e^{3x}(x+1)^2}$$

13) Solve

a. $\frac{x+3}{x-1} \leq 4$

$$\frac{x+3}{x-1} - 4 \leq 0$$

$$\frac{x+3-4(x-1)}{x-1} \leq 0$$

$$\frac{-3x+7}{x-1} \leq 0$$

c.v. $x = 7/3$ and 1

1	7/3
-	+
0	0

$$(-\infty, 1) \cup [7/3, \infty)$$

b. $\frac{5x(x+1)+10(x+1)^2}{20(x+2)(x+1)^3} \leq 0$

$$\frac{5(x+1)[x+2(x+1)]}{4 \cdot 20(x+2)(x+1)^3} \leq 0$$

$$\frac{3x+2}{4(x+2)(x+1)^2} \leq 0$$

c.v. $x = -2/3, -2, -1$

-2	-1	-2/3
+	-	-
0	0	0

$$(-2, -1) \cup (-1, -2/3)$$

FUNCTION OPERATIONS AND COMPOSITION; INVERSE FUNCTIONS

Composition of a function g with a function f is defined as:

$$h(x) = g(f(x))$$

The domain of h is the set of all x -values such that x is in the domain of f and $f(x)$ is in the domain of g .

Inverses: Functions f and g are inverses of each other provided:

$$f(g(x)) = x \text{ and } g(f(x)) = x$$

The function g is denoted as f^{-1} , read as "f inverse" and $f^{-1}(x) \neq \frac{1}{f(x)}$

Horizontal Line Test: If any horizontal line which is drawn through the graph of a function f intersects the graph no more than once, then f is said to be a **one-to-one** function and has an inverse.

14) Let f and g be functions whose values are given by the table below. Assume g is one-to-one.

a. $f(g(3)) = f(4) = -1$

b. $g^{-1}(4) = g(x) : (3, 4) \leftrightarrow g^{-1}(x) = (4, 3)$
 $\therefore g^{-1}(4) = 3$

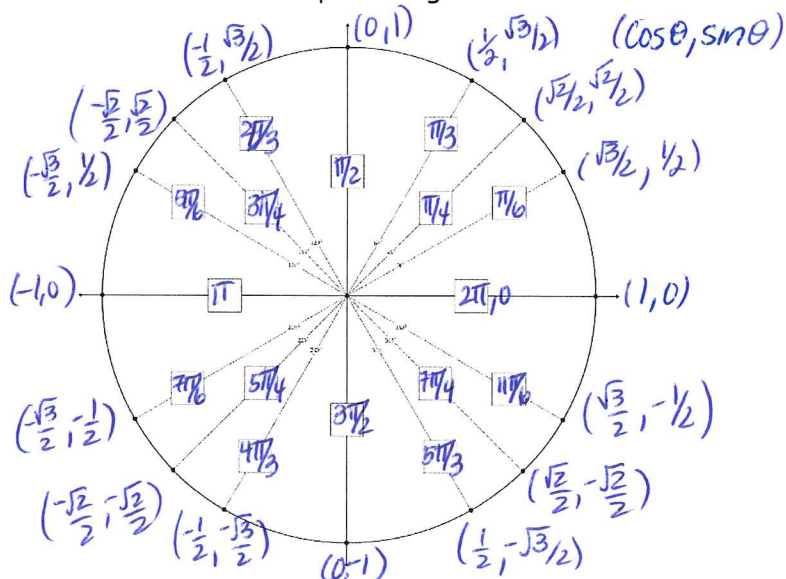
c. $f(g^{-1}(6)) = f(4) = -1$

d. $f^{-1}(f(g(2))) = f^{-1}(f(3)) = f^{-1}(10) = 3$

x	$f(x)$	$g(x)$
1	6	2
2	9	3
3	10	4
4	-1	6

TRIGONOMETRY

UNIT CIRCLE: Complete angles and Sine/Cosine



Key Trig Identities - formulas you should know:

$$\sin^2 x + \cos^2 x = 1$$

$$\sin(2x) = 2 \sin x \cos x$$

$$\cos(2x) = 1 - 2 \sin^2 x$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin(-x) = -\sin x \text{ ODD}$$

$$\cos(-x) = \cos x \text{ EVEN}$$

15) Without using a calculator or table, find each value:

a. $\cos\left(-\frac{\pi}{3}\right)$

$$\frac{1}{2}$$

b. $\sec\left(\frac{11\pi}{6}\right)$

$$\frac{2\sqrt{3}}{3}$$

c. $\tan^{-1}(-1)$

$$-\frac{\pi}{4}$$

d. $\sec^{-1}(-2)$

$$\frac{2\pi}{3}$$

16) Solve the trigonometric equations algebraically by using identities and **without** the use of a calculator. Find all solutions in the interval $0 \leq \theta \leq 2\pi$.

a. $2\sin^2\theta - 1 = 0$

$$2\sin^2\theta = 1$$

$$\sin^2\theta = \frac{1}{2}$$

$$\sin\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$

$$\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$$

b. $\sin(2x) = \cos x$

$$2\sin x \cos x = \cos x$$

$$2\sin x \cos x - \cos x = 0$$

$$\cos x(2\sin x - 1) = 0$$

↓

↓

$$x = \pi/2, 3\pi/2, \pi/6, 5\pi/6$$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS:

Logarithm: For any positive numbers b and y with $b \neq 1$, we define the **logarithm of y with base b** as follows: $\log_b y = x$ if and only if $b^x = y$

LAWS OF LOGARITHMS (for $M, N, b > 0, b \neq 1$)

(i) $\log_b MN = \log_b M + \log_b N$

(ii) $\log_b \frac{M}{N} = \log_b M - \log_b N$

(iii) $\log_b M = \log_b N$ if and only if $M = N$

(iv) $\log_b M^k = k \cdot \log_b M$

(v) $b^{\log_b M} = M$

The logarithmic and exponential functions are **inverse** functions.

Example: Consider $f(x) = 2^x$ and $g(x) = \log_2 x$. Verify that (3,8) is on the graph of f and (8,3) on the graph of g . In addition, $f(g(x)) = 2^{\log_2 x} = x$ and $g(f(x)) = \log_2(2^x) = x$ which verifies that f and g are indeed inverse functions.

Simplify.

17) $\log_5 \frac{1}{125} = \log_5 5^{-3} = -3$

18) $\log_4 \sqrt[8]{16} = \log_4 (4^2)^{1/8} = \log_4 4^{2/8} = 1/4$

19) $2 \ln \frac{3}{e^3} = \ln \left(\frac{3}{e^3}\right)^2 = \ln \left(\frac{9}{e^6}\right) = \ln 9 - \ln e^6 = \ln 9 - 6$ or $2 \ln(3) - 6$

20) $\frac{1}{t} \ln e^t = \frac{\ln e^t}{t} = \frac{t}{t} = 1$

21) $e^{3 \ln(x+5)} = e^{\ln(x+5)^3} = (x+5)^3$

22) $e^{-\ln x} = e^{\ln(x)^{-1}} = x^{-1}$ or $\frac{1}{x}$

23) $4 \ln e^{x^2} = 4x^2$

24) $e^{3x + \ln 5} = e^{3x} \cdot e^{\ln 5} = 5e^{3x}$

25) $e^{3(\ln(x) + \ln 5)} = e^{3(\ln(5x))} = e^{\ln(5x)^3} = (5x)^3 = 125x^3$

26) Write $5 \ln(x) - 3 \ln(y) + 2 \ln(4x) - 6 \ln(y^2)$ as a single logarithm.

$$\ln x^5 - \ln y^3 + \ln 16x^2 - \ln y^{12} = \ln \left(\frac{x^5 \cdot 16x^2}{y^3 \cdot y^{12}} \right) = \ln \left(\frac{16x^7}{y^{15}} \right)$$

27) Expand $\ln \left(\frac{3x^2}{2\sqrt{y^2-4}} \right)$.

$$= \ln 3 + \ln x^2 - \ln 2 - \ln (y^2-4)^{1/2} = \ln 3 + 2 \ln x - \ln 2 - \frac{1}{2} \ln (y^2-4)$$

28) Evaluate:

a. $\ln 1$

0

b. $\ln 3e$

$\ln 3 + \ln e$
 $\ln(3) + 1$

c. $\ln e$

1

d. $\ln 0$

DNE

e. $\ln(\ln e)$

$\ln(1) = 0$

Limits: $\lim_{x \rightarrow c} f(x) = N$ "Limit as x approaches c of $f(x)$ equals N "

LIMIT \neq CONTINUITY

Understanding the difference between limits and continuity: Limit is the value (y) that the function APPROACHES as you get close to c (either from one side or from both sides). Continuity implies that the limit exists (from both sides) and that the limit = the value at c !

Limit Exists: $\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) \therefore \lim_{x \rightarrow c} f(x)$ exists and will equal all the same value or $\pm \infty$.

Continuous: $\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) = f(c) = A$ (where A is a constant not $\pm \infty$) **MUST SHOW ALL 3!!**

One-Sided limit: exists if graph approaches a value from one side: + (right) or - (left)

$$\lim_{x \rightarrow a} C = C$$

Evaluating Limits:

- Tables:** either from table or by graphing and viewing table looking for values of y approaching the same number. Check one sided limits first and if they are equal the overall limit exists.
- Graphically:** Check one sided limits (again - value of limit is the y -value or the height of the graph) and if they are equal overall limit exists.
- Algebraically - Direct substitution:** plug in c . If you get a constant then limit exists. Rational functions may need to be simplified first!

When you use Direct substitution and you get

- $\frac{0}{\text{number}}$ then the limit is equal to 0
- $\frac{0}{0}$ most likely a removable discontinuity, try simplifying or rationalizing.
- $\frac{\text{number}}{0}$ most likely an infinite discontinuity, look at each side and evaluate one sided limit or if asking for two sided limit make sure the function is approaching the same from both sides.

$\lim_{x \rightarrow \pm \infty} f(x)$: **End Behavior:**

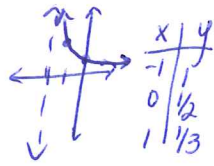
- Polynomial: use degree (even or odd) and sign of leading coefficient to decide parabolic or cubic.
- Rational: end behavior is determined by the Horizontal or Oblique Asymptotes.
- Other "Known" Graphs: know the graphs!

Examples: Evaluate the limits:

29) $\lim_{x \rightarrow 3} (2x^2)$

$2(3)^2 = 2(9) = 18$

30) $\lim_{x \rightarrow -2^+} \frac{1}{x+2} = \frac{1}{0} = \frac{\#}{0}$ VA!

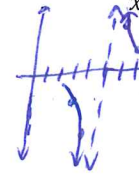


$\lim_{x \rightarrow -2^+} \frac{1}{x+2} = \infty$

31) $\lim_{x \rightarrow \frac{\pi}{2}} (x \sin x)$

$= \frac{\pi}{2} \sin(\frac{\pi}{2}) = \frac{\pi}{2}(1) = \frac{\pi}{2}$

32) $\lim_{x \rightarrow 5} \frac{3}{x-5} = \frac{3}{0} = \frac{\#}{0}$ VA



both sides
 $\lim_{x \rightarrow 5^-} \frac{3}{x-5} = -\infty$
 $\lim_{x \rightarrow 5^+} \frac{3}{x-5} = \infty$

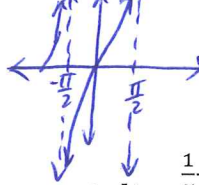
33) $\lim_{x \rightarrow 0^+} \ln(x) = -\infty$

"Known" graph



34) $\lim_{x \rightarrow \frac{\pi}{2}} \tan(x)$

"known" graph



= DNE b/c
 $\lim_{x \rightarrow \frac{\pi}{2}^-} \tan(x) = \infty$
 $\lim_{x \rightarrow \frac{\pi}{2}^+} \tan(x) = -\infty$

35) $\lim_{x \rightarrow -\infty} 3e^{2x}$

E.B. → "known" graph



$\therefore = 0$

36) $\lim_{x \rightarrow 1} \frac{3x^2 - 2x - 1}{x^2 - 1}$ $\lim_{x \rightarrow 5} \frac{3}{x-5} = \frac{\#}{0}$ VA

$= \frac{3-2-1}{1-1} = \frac{0}{0}$ removable

$= \lim_{x \rightarrow 1} \frac{(3x+1)(x-1)}{(x+1)(x-1)}$
 $= \frac{3+1}{1+1} = \frac{4}{2} = 2$

37) $\lim_{x \rightarrow \infty} \frac{2x}{x-1}$

E.B. → HA: $y = \frac{2}{1} = 2$

$\therefore \lim_{x \rightarrow \infty} \frac{2x}{x-1} = 2$

38) $\lim_{x \rightarrow 1} \frac{x-1}{x-1}$

$= \frac{1-1}{1-1} = \frac{0}{0}$ rem. E.B.

$\lim_{x \rightarrow 1} \frac{(x-1)x}{(x-1)x} = \lim_{x \rightarrow 1} \frac{1-x}{(x-1)(x)}$
 $= \lim_{x \rightarrow 1} \frac{-(x-1)}{(x-1)x} = \frac{-1}{1} = -1$

39) $\lim_{x \rightarrow -\infty} 2x(x-3)^2(x+1)$

leading term: $2x(x^2)x = 2x^4$

$\lim_{x \rightarrow -\infty} 2x(x-3)^2(x+1) = \infty$

40) $\lim_{x \rightarrow -\infty} \frac{3}{x-5}$

E.B. → HA: $y = 0$

$\lim_{x \rightarrow -\infty} \frac{3}{x-5} = 0$

41) If $f(x) = \begin{cases} 2x-3, & \text{if } x < -1 \\ 2, & \text{if } x = -1 \\ 5x, & \text{if } x > -1 \end{cases}$, determine the following:

a. $\lim_{x \rightarrow -1^-} f(x)$
 $\rightarrow x < -1$

$= 2(-1) - 3 = -2 - 3 = -5$

b. $\lim_{x \rightarrow -1^+} f(x)$
 $\rightarrow x > -1$

$= 5(-1) = -5$

c. $\lim_{x \rightarrow -1} f(x) = -5$

b/c $\lim_{x \rightarrow -1^-} f(x) = -5$
 $\lim_{x \rightarrow -1^+} f(x) = -5$

d. Is f continuous?

No, b/c

$\lim_{x \rightarrow -1} f(x) = -5 \neq f(-1) = 2$

42) If $f(x) = |x+2| = \begin{cases} x+2, & x \geq -2 \\ -(x+2), & x < -2 \end{cases}$, determine the following:

a. $\lim_{x \rightarrow -2^-} f(x)$
 $\rightarrow x < -2$

$\lim_{x \rightarrow -2^-} f(x) = -(-2+2) = 0$

b. $\lim_{x \rightarrow -2^+} f(x)$
 $\rightarrow x > -2$

$\lim_{x \rightarrow -2^+} f(x) = -(-2+2) = 0$

c. $\lim_{x \rightarrow -2} f(x) = 0$

b/c
 $\lim_{x \rightarrow -2^-} f(x) = 0$
 $\lim_{x \rightarrow -2^+} f(x) = 0$

d. Is f continuous?

Yes, b/c

$\lim_{x \rightarrow -2} f(x) = f(-2) = 0$

43) Determine whether $f(x)$ is continuous. Justify. Then, determine domain, interval of continuity and location and type of any discontinuities.

a. $f(x) = \begin{cases} 3x^2, & x < 0 \\ 4, & x = 0 \\ 3 \sin^2 x, & x > 0 \end{cases}$

$\lim_{x \rightarrow 0^-} f(x) = 3(0)^2 = 0 \therefore \lim_{x \rightarrow 0} f(x) = 0$

$\lim_{x \rightarrow 0^+} f(x) = 3(\sin 0)^2 = 0$

$f(0) = 4 \therefore$ discontinuous at $x=0$ removable w/ pt

b. $f(x) = \begin{cases} \ln x, & x < e^2 \\ \sqrt{x}, & x \geq e^2 \end{cases}$

$\lim_{x \rightarrow e^{2-}} f(x) = \ln e^2 = 2$

$\lim_{x \rightarrow e^{2+}} f(x) = f(e^2) = \sqrt{e^2} = e$

$2 \neq e \therefore$ discontin. at $x = e^2$ jump.

c. $f(x) = \begin{cases} 3x^2 - 1, & x \leq 2 \\ 5x + 1, & x > 2 \end{cases}$

$\lim_{x \rightarrow 2} f(x) = f(2) = 3(2)^2 - 1 = 12 - 1 = 11$

$\lim_{x \rightarrow 2^+} f(x) = 5(2) + 1 = 11$

$\therefore \lim_{x \rightarrow 2} f(x) = f(2) = 11$ cont.!

44) Let $f(x)$ be the graph below. Determine the following:

a. $\lim_{x \rightarrow -3^-} f(x) = -\infty$

b. $\lim_{x \rightarrow -3^+} f(x) = -\infty$

c. $\lim_{x \rightarrow -3} f(x) = -\infty$

d. $f(-3) = \text{und.}$

e. Is f continuous at -3 ? *NO, infinite discontinuity*

f. $\lim_{x \rightarrow -1^-} f(x) = 4$

g. $\lim_{x \rightarrow -1^+} f(x) = 4$

h. $\lim_{x \rightarrow -1} f(x) = 4$

i. $f(-1) = 2$

j. $\lim_{x \rightarrow 1^-} f(x) = -\infty$

k. $\lim_{x \rightarrow 1^+} f(x) = \infty$

l. $\lim_{x \rightarrow 1} f(x) = \text{DNE b/c } -\infty \neq \infty$

m. $f(1) = \text{und.}$

n. $\lim_{x \rightarrow 3^-} f(x) = 1$

o. $\lim_{x \rightarrow 3^+} f(x) = -3$

p. $\lim_{x \rightarrow 3} f(x) = \text{DNE b/c } 1 \neq -3$

q. $f(3) = 1$

r. $\lim_{x \rightarrow -\infty} f(x) = \infty$

s. $\lim_{x \rightarrow \infty} f(x) = -3$

t. Continuity Interval: $(-\infty, -3) \cup (-3, -1) \cup (-1, 1) \cup (1, 3) \cup (3, \infty)$

u. Location and type of each discontinuity:

$x = -3$ infinite

$x = 1$ infinite

$x = -1$ removable pt

$x = 3$ jump

v. Domain: $(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$

45) Draw a graph given the following conditions:

◆ $f(0) = 0$

◆ $f(1) = 2$

◆ $f(4) = -3$

◆ $\lim_{x \rightarrow 3^-} f(x) = \infty$

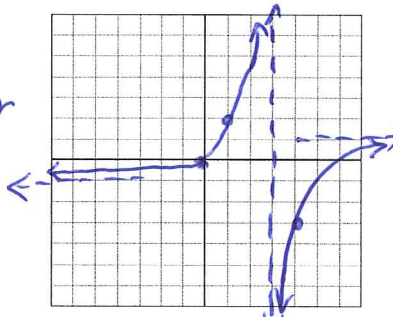
VA at $x=3$

◆ $\lim_{x \rightarrow -\infty} f(x) = -1$

◆ $\lim_{x \rightarrow \infty} f(x) = 1$

◆ $\lim_{x \rightarrow 3^+} f(x) = -\infty$

End behavior



46) Draw a graph given the following conditions:

◆ $f(-2) = 0$

◆ $f(2) = 2$

◆ $\lim_{x \rightarrow -2^-} f(x) = 4$

← jump →

◆ $\lim_{x \rightarrow -\infty} f(x) = -\infty$

◆ $\lim_{x \rightarrow \infty} f(x) = 5$

◆ $\lim_{x \rightarrow -2^+} f(x) = 0$

End behavior

